Lifshitz Tails in Constant Magnetic Fields

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Abstract: We consider the 2D Landau Hamiltonian H perturbed by a random alloy-type potential, and investigate the Lifshitz tails, i.e. the asymptotic behavior of the corresponding integrated density of states (IDS) near the edges in the spectrum of H. If a given edge coincides with a Landau level, we obtain different asymptotic formulae for power-like, exponential sub-Gaussian, and super-Gaussian decay of the one-site potential. If the edge is away from the Landau levels, we impose a rational-flux assumption on the magnetic field, consider compactly supported one-site potentials, and formulate a theorem which is analogous to a result obtained by the first author and T. Wolff in [25] in the case of a vanishing magnetic field.

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1 Introduction

Let

$$H_0 = H_0(b) := (-i\nabla - A)^2 - b \tag{1.1}$$

be the unperturbed Landau Hamiltonian, essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. Here $A = (-\frac{bx_2}{2}, \frac{bx_1}{2})$ is the magnetic potential, and $b \geq 0$ is the constant scalar magnetic field. It is well-known that if $b > 0$, then the spectrum $\sigma(H_0)$ of the operator $H_0(b)$ consists of the so-called Landau levels $2bq$, $q \in \mathbb{Z}_+$, and each Landau level is an eigenvalue of infinite multiplicity. If $b = 0$, then $H_0 = -\Delta$, and $\sigma(H_0) = [0, \infty)$ is absolutely continuous. Next, we introduce a random $\mathbb{Z}^2$-ergodic alloy-type electric potential

$$V(x) = V_\omega(x) := \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma u(x - \gamma), \quad x \in \mathbb{R}^2.$$

Our general assumptions concerning the potential V_ω are the following ones:

- $\mathbf{H}_1$: The single-site potential u satisfies the estimates

$$0 \leq u(x) \leq C_0(1 + |x|)^{-\varkappa}, \quad x \in \mathbb{R}^2, \tag{1.2}$$

with some $\varkappa > 2$ and $C_0 > 0$. Moreover, there exists an open non-empty set $\Lambda \subset \mathbb{R}^2$ and a constant $C_1 > 0$ such that $u(x) \geq C_1$ for $x \in \Lambda$.

- $\mathbf{H}_2$: The coupling constants $\{\omega_\gamma\}_{\gamma\in\mathbb{Z}^2}$ are non-trivial, almost surely bounded i. i. d. random variables.

Evidently, these two assumptions entail

$$M := \operatorname*{ess-sup}_{\omega} \sup_{x\in\mathbb{R}^2} |V_\omega(x)| < \infty. \tag{1.3}$$

On the domain of H_0 define the operator $H = H_\omega := H_0(b) + V_\omega$. The integrated density of states (IDS) for the operator H is defined as a non-decreasing left-continuous function $\mathcal{N}_b : \mathbb{R} \to [0,\infty)$ which almost surely satisfies

$$\int_{\mathbb{R}} \varphi(E) d\mathcal{N}_b(E) = \lim_{R\to\infty} R^{-2} \mathrm{Tr}\, \left(\mathbf{1}_{\Lambda_R} \varphi(H) \mathbf{1}_{\Lambda_R}\right), \quad \forall \varphi \in C_0^\infty(\mathbb{R}). \tag{1.4}$$

Here and in the sequel $\mathbf{1}_{\mathcal{O}}$ denotes the the characteristic function of the set $\mathcal{O}$, and $\Lambda_R := \left(-\frac{R}{2}, \frac{R}{2}\right)^2$. By the Pastur-Shubin formula (see e.g. [36, Section 2] or [11, Corollary 3.3]) we have

$$\int_{\mathbb{R}} \varphi(E) d\mathcal{N}_b(E) = \mathbb{E}\left(\mathrm{Tr}\, \left(\mathbf{1}_{\Lambda_1} \varphi(H) \mathbf{1}_{\Lambda_1}\right)\right), \quad \forall \varphi \in C_0^\infty(\mathbb{R}), \tag{1.5}$$

where $\mathbb{E}$ denotes the mathematical expectation. Moreover, there exists a set $\Sigma \subset \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ almost surely, and supp $d\mathcal{N}_b = \Sigma$. The aim of the present article is to study the asymptotic behavior of $\mathcal{N}_b$ near the edges of Σ. It is well known that, for many random models, this behavior is characterized by a very fast decay which goes under the name of "Lifshitz tails". It was studied extensively in the absence of magnetic field (see e.g. [31], [15]), and also in the presence of magnetic field for other types of disorder (see [2], [6], [12], [7], [13]).

2 Main results

In order to fix the picture of the almost sure spectrum $\sigma(H_\omega)$, we assume $b > 0$, and make the following two additional hypotheses:

- $\mathbf{H}_3$: The support of the random variables ω_γ, $\gamma \in \mathbb{Z}^2$, consists of the interval $[\omega_-, \omega_+]$ with $\omega_- < \omega_+$ and $\omega_-\omega_+ \le 0$.
- $\mathbf{H}_4$: We have $M_+ - M_- < 2b$ where $\pm M_\pm := \text{ess-sup}_\omega \sup_{x\in\mathbb{R}^2} (\pm V_\omega(x))$.

Assumptions $\mathbf{H}_1 - \mathbf{H}_4$ imply $M_- M_+ \le 0$. Moreover, the union $\cup_{q=0}^\infty [2bq + M_-, 2bq + M_+]$ which contains Σ, is disjoint. Introduce the bounded $\mathbb{Z}^2$-periodic potential

$$W(x) := \sum_{\gamma\in\mathbb{Z}^2} u(x-\gamma), \quad x \in \mathbb{R}^2,$$

and on the domain of H_0 define the operators $H^{\pm} := H_0 + \omega_{\pm} W$. It is easy to see that

$$\sigma(H^-) \subseteq \cup_{q=0}^{\infty}[2bq + M_-, 2bq], \quad \sigma(H^+) \subseteq \cup_{q=0}^{\infty}[2bq, 2bq + M_+],$$

and

$$\sigma(H^-) \cap [2bq + M_-, 2bq] \neq \emptyset, \quad \sigma(H^+) \cap [2bq, 2bq + M_+] \neq \emptyset, \quad \forall q \in \mathbb{Z}_+.$$

Set

$$E_q^- := \inf \left\{ \sigma(H^-) \cap [2bq + M_-, 2bq] \right\}, \quad E_q^+ := \sup \left\{ \sigma(H^+) \cap [2bq, 2bq + M_+] \right\}.$$

Following the argument in [16] (see also [31, Theorem 5.35]), we easily find that

$$\Sigma = \cup_{q=0}^{\infty}[E_q^-, E_q^+],$$

i.e. Σ is represented as a disjoint union of compact intervals, and each interval $[E_q^-, E_q^+]$ contains exactly one Landau level $2bq$, $q \in \mathbb{Z}_+$.

In the following theorems we describe the behavior of the integrated density of states $\mathcal{N}_b$ near E_q^-, $q \in \mathbb{Z}_+$; its behavior near E_q^+ could be analyzed in a completely analogous manner.

Our first theorem concerns the case where $E_q^- = 2bq$, $q \in \mathbb{Z}_+$. This is the case if and only if $\omega_- = 0$; in this case, the random variables ω_γ, $\gamma \in \mathbb{Z}^2$, are non-negative.

Theorem 2.1. *Let $b > 0$ and assumptions $\mathbf{H}_1$ – $\mathbf{H}_4$ hold. Suppose that $\omega_- = 0$, and that*

$$\mathbb{P}(\omega_0 \leq E) \sim C E^{\kappa}, \quad E \downarrow 0, \tag{2.1}$$

for some $C > 0$ and $\kappa > 0$. Fix the Landau level $2bq = E_q^-$, $q \in \mathbb{Z}_+$.
i) Assume that $C_-(1 + |x|)^{-\varkappa} \leq u(x) \leq C_+(1 + |x|)^{-\varkappa}$, $x \in \mathbb{R}^2$, for some $\varkappa > 2$, and $C_+ \geq C_- > 0$. Then we have

$$\lim_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq))|}{\ln E} = -\frac{2}{\varkappa - 2}. \tag{2.2}$$

ii) Assume $\frac{e^{-C_+|x|^{\beta}}}{C_+} \leq u(x) \leq \frac{e^{-C_-|x|^{\beta}}}{C_-}$, $x \in \mathbb{R}^2$, $\beta \in (0, 2]$, $C_+ \geq C_- > 0$. Then we have

$$\lim_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} = 1 + \frac{2}{\beta}. \tag{2.3}$$

iii) Assume $\frac{\mathbf{1}_{\{x \in \mathbb{R}^2 \, ; \, |x - x_0| < \varepsilon\}}}{C_+} \leq u(x) \leq \frac{e^{-C_-|x|^2}}{C_-}$ for some $C_+ \geq C_- > 0$, $x_0 \in \mathbb{R}^2$, and $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$1 + \delta \leq \liminf_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq)|}{\ln |\ln E|} \leq$$

$$\limsup_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq)|}{\ln |\ln E|} \leq 2. \tag{2.4}$$

The proof of Theorem 2.1 is contained in Sections 3 – 5. In Section 3 we construct a periodic approximation of the IDS $\mathcal{N}_b$ which plays a crucial role in this proof. The upper bounds of the IDS needed for the proof of Theorem 2.1 are obtained in Section 4, and the corresponding lower bounds are deduced in Section 5.

Remarks: i) In the first and second part of Theorem 2.1 we consider one-site potentials u respectively of power-like or exponential sub-Gaussian decay at infinity, and obtain the values of the so called Lifshitz exponents. Note however that in the case of power-like decay of u the double logarithm of $\mathcal{N}_b(2bq+E)-\mathcal{N}(2bq)$ is asymptotically proportional to $\ln E$ (see (2.2)), while in the case of exponentially decaying u this double logarithm is asymptotically proportional to $\ln|\ln E|$ (see (2.3)); in both cases the Lifshitz exponent is defined as the corresponding proportionality factor. In the third part of the theorem which deals with one-site potentials u of super-Gaussian decay, we obtain only upper and lower bounds of the Lifshitz exponent. It is natural to conjecture that the value of this exponent is 2, i.e. that the upper bound in (2.4) reveals the correct asymptotic behavior.

ii) In the case of a vanishing magnetic field, the Lifshitz asymptotics for random Schrödinger operator with repulsive random alloy-type potentials has been known since long ago (see [17]). To the authors' best knowledge the Lifshitz asymptotics for the Landau Hamiltonian with non-zero magnetic field, perturbed by a positive random alloy-type potential, is considered for the first time in the present article. However, it is appropriate to mention here the related results concerning the Landau Hamiltonian with repulsive random Poisson potential. In [2] the Lifshitz asymptotics in the case of a power-like decay of the one-site potential u, was investigated. The case of a compact support of u was considered in [6]. The results for the case of a compact support of u were essentially used in [12] and [7] (see also [13]), in order to study the problem in the case of an exponential decay of u.

Our second theorem concerns the case where $E_q^- < 2bq$, $q \in \mathbb{Z}_+$. This is the case if and only if $\omega_- < 0$. In order to handle this case, we need some facts from the magnetic Floquet-Bloch theory. Let $\Gamma := g_1\mathbb{Z} \oplus g_2\mathbb{Z}$ with $g_j > 0$, $j = 1, 2$. Introduce the tori

$$\mathbb{T}_\Gamma := \mathbb{R}^2/\Gamma, \quad \mathbb{T}^*_\Gamma := \mathbb{R}^2/\Gamma^*, \tag{2.5}$$

where $\Gamma^* := 2\pi g_1^{-1}\mathbb{Z} \oplus 2\pi g_2^{-1}\mathbb{Z}$ is the lattice dual to Γ. Denote by $\mathcal{O}_\Gamma$ and $\mathcal{O}^*_\Gamma$ the fundamental domains of $\mathbb{T}_\Gamma$ and $\mathbb{T}^*_\Gamma$ respectively. Let $\mathcal{W} : \mathbb{R}^2 \to \mathbb{R}$ be a Γ-periodic bounded real-valued function. On the domain of H_0 define the operator $H_\mathcal{W} := H_0+\mathcal{W}$. Assume that the scalar magnetic field $b \geq 0$ satisfies *the integer-flux* condition with respect to the lattice Γ, i.e. that $bg_1g_2 \in 2\pi\mathbb{Z}_+$. Fix $\theta \in \mathbb{T}^*_\Gamma$. Denote by $h_0(\theta)$ the self-adjoint operator generated in $L^2(\mathcal{O}_\Gamma)$ by the closure of the non-negative quadratic form

$$\int_{\mathcal{O}_\Gamma} |(i\nabla + A - \theta)f|^2 dx$$

defined originally on the set

$$\left\{ f = g_{\big|\mathcal{O}_\Gamma} \mid g \in C^\infty(\mathbb{R}^2),\, (\tau_\gamma g)(x) = g(x),\, x \in \mathbb{R}^2,\, \gamma \in \Gamma \right\}$$

where τ_y, $y \in \mathbb{R}^2$, is the magnetic translation given by

$$(\tau_y g)(x) := e^{ib\frac{y_1 y_2}{2}} e^{ib\frac{x\wedge y}{2}} g(x+y), \quad x \in \mathbb{R}^2, \tag{2.6}$$

with $x \wedge y := x_1 y_2 - x_2 y_1$. Note that the integer-flux condition implies that the operators τ_γ, $\gamma \in \Gamma$, commute with each other, as well as with operators $i\frac{\partial}{\partial x_j} + A_j$, $j = 1, 2$ (see (1.1)), and hence with H_0 and $H_{\mathcal{W}}$. In the case $b = 0$, the domain of the operator h_0 is isomorphic to the Sobolev space $\mathrm{H}^2(\mathbb{T}_\Gamma)$, but if $b > 0$, this is not the case even under the integer-flux assumption since h_0 acts on $U(1)$-sections rather than on functions over $\mathbb{T}_\Gamma$ (see e.g [30, Subsection 2.2]). On the domain of h_0 define the operator

$$h_{\mathcal{W}}(\theta) := h_0(\theta) + \mathcal{W}, \quad \theta \in \mathbb{T}_\Gamma^*. \tag{2.7}$$

Set

$$\mathcal{H}_0 := \int_{\mathcal{O}_\Gamma^*} \oplus\, h_0(\theta) d\theta, \quad \mathcal{H}_{\mathcal{W}} := \int_{\mathcal{O}_\Gamma^*} \oplus\, h_{\mathcal{W}}(\theta) d\theta. \tag{2.8}$$

It is well-known (see e.g [10], [35], or [30, Subsection 2.4]) that the operators H_0 and $H_{\mathcal{W}}$ are unitarily equivalent to the operators $\mathcal{H}_0$ and $\mathcal{H}_{\mathcal{W}}$ respectively. More precisely, we have $H_0 = U^* \mathcal{H}_0 U$ and $H_{\mathcal{W}} = U^* \mathcal{H}_{\mathcal{W}} U$ where $U : L^2(\mathbb{R}^2) \to L^2(\mathcal{O}_\Gamma \times \mathcal{O}_\Gamma^*)$ is the unitary Gelfand-type operator defined by

$$(Uf)(x;\theta) := \frac{1}{\sqrt{\operatorname{vol} \mathbb{T}_\Gamma^*}} \sum_{\gamma \in \Gamma} e^{-i\theta(x+\gamma)} (\tau_\gamma f)(x), \quad x \in \mathcal{O}_\Gamma, \quad \theta \in \mathbb{T}_\Gamma^*. \tag{2.9}$$

Evidently for each $\theta \in \mathbb{T}_\Gamma^*$ the spectrum of the operator $h_{\mathcal{W}}(\theta)$ is purely discrete. Denote by $\{E_j(\theta)\}_{j=1}^\infty$ the non-decreasing sequence of its eigenvalues. Let $E \in \mathbb{R}$. Set

$$J(E) := \{ j \in \mathbb{N}\,;\, \text{there exists } \theta \in \mathbb{T}_\Gamma^* \text{ such that } E_j(\theta) = E \}.$$

Evidently, for each $E \in \mathbb{R}$ the set $J(E)$ is finite. If $E \in \mathbb{R}$ is an end of an open gap in $\sigma(H_0 + \mathcal{W})$, then we will call it an edge in $\sigma(H_0 + \mathcal{W})$. We will call the edge E in $\sigma(H_0 + \mathcal{W})$ *simple* if $\#J(E) = 1$. Moreover, we will call the edge E *non-degenerate* if for each $j \in J(E)$ the number of points $\theta \in \mathbb{T}_\Gamma^*$ such that $E_j(\theta) = E$ is finite, and at each of these points the extremum of E_j is non-degenerate.
Assume at first that $b = 0$. Then $H_0 = -\Delta$, and we will consider the general d-dimensional situation; the simple and non-degenerate edges in $\sigma(-\Delta + \mathcal{W})$ are defined exactly as in the two-dimensional case. If $\mathcal{W} : \mathbb{R}^d \to \mathbb{R}$ is a real-valued bounded periodic function, it is well-known that:

- The spectrum of $-\Delta+\mathcal{W}$ is absolutely continuous (see e.g. [33, Theorems XIII.90, XIII.100]). In particular, no Floquet eigenvalue $E_j : \mathbb{T}^*_\Gamma \to \mathbb{R}$, $j \in \mathbb{N}$, is constant.

- If $d = 1$, all the edges in $\sigma(-\Delta + \mathcal{W})$ are simple and non-degenerate (see e.g. [33, Theorem XIII.89]).

- For $d \geq 1$ the bottom of the spectrum of $-\Delta + \mathcal{W}$ is a simple and non-degenerate edge (see [19]).

- For $d \geq 1$, the edges of $\sigma(-\Delta + \mathcal{W})$ generically are simple (see [24]).

Despite the widely spread belief that generically the higher edges in $\sigma(-\Delta+\mathcal{W})$ should also be non-degenerate in the multi-dimensional case $d > 1$, there are no rigorous results in support of this conjecture.
Let us go back to the investigation of the Lifshitz tails for the operator $-\Delta + V_\omega$. It follows from the general results of [16] that E^- (respectively, E^+) is an upper (respectively, lower) end of an open gap in $\sigma(-\Delta+V_\omega)$ if and only if it is an upper (respectively, lower) end of an open gap in the spectrum of $-\Delta + \omega_- W$ (respectively, $-\Delta + \omega_+ W$). For definiteness, let us consider the case of an upper end E^-. The asymptotic behavior of the IDS $\mathcal{N}_0(E)$ as $E \downarrow E^-$ has been investigated in [28] - [29] in the case $d = 1$, and in [19] in the case $d \geq 1$ and $E^- = \inf \sigma(-\Delta + \omega_- W)$. Note that the proofs of the results of [28], [29], and [19], essentially rely on the non-degeneracy of E^-. Later, the Lifshitz tails for the operator $-\Delta + V_\omega$ near the edge E^- were investigated in [15] under the assumptions that $d \geq 1$, $E^- > \inf \sigma(-\Delta + \omega_- W)$, and that E^- is non-degenerate edge in the spectrum of $-\Delta + \omega_- W$; due to the last assumption these results are conditional. However, it turned out possible to lift the non-degeneracy assumption in the two-dimensional case considered in [25]. First, it was shown in [25, Theorem 0.1] that for any single-site potential u satisfying assumption $\mathbf{H}_1$, we have

$$\limsup_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_0(E^- + E) - \mathcal{N}_0(E_q^-))|}{\ln E} < 0$$

without any additional assumption on E^-. If, moreover, the support of u is compact, and the probability $\mathbb{P}(\omega_0 - \omega_- \leq E)$ admits a power-like decay as $E \downarrow 0$, it follows from [25, Theorem 0.2] that there exists $\alpha > 0$ such that

$$\lim_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_0(E^- + E) - \mathcal{N}_0(E_q^-))|}{\ln E} = -\alpha \tag{2.10}$$

under the unique generic hypothesis that E^- is a simple edge. Note that the absolute continuity of $\sigma(-\Delta + \omega_- W)$ plays a crucial role in the proofs of the results of [25].
Assume now that the scalar magnetic field $b > 0$ satisfies the rational flux condition $b \in 2\pi\mathbb{Q}$. More precisely, we assume that $b/2\pi$ is equal to the irreducible fraction p/r, $p \in \mathbb{N}$, $r \in \mathbb{N}$. Then b satisfies the integer-flux assumption with respect, say, to the lattice $\Gamma = r\mathbb{Z} \oplus \mathbb{Z}$, and the operator H^- is unitarily equivalent to $\mathcal{H}_{\omega_- W}$. As in the

non-magnetic case, in order to investigate the Lifshitz asymptotics as $E \downarrow E_q^-$ of $\mathcal{N}_b(E)$, we need some information about the character of E_q^- as an edge in the spectrum of H^-. For example, if we assume that E_q^- is a simple edge, and the corresponding Floquet band does not shrink into a point, we can repeat almost word by word the argument of the proof of [25, Theorem 0.2], and obtain the following

Theorem 2.2. *Let $b > 0$, $b \in 2\pi\mathbb{Q}$, and assumptions $\mathbf{H}_1$ – $\mathbf{H}_4$ hold. Assume that the support of u is compact, $\omega_- < 0$, and $\mathbb{P}(\omega_0 - \omega_- \leq E) \sim CE^\kappa$, $E \downarrow 0$, for some $C > 0$ and $\kappa > 0$. Fix $q \in \mathbb{Z}_+$. Suppose E_q^- is a simple edge in the spectrum of the operator H^-, and that the function E_j, $j \in J(E_q^-)$, is not identically constant. Then there exists $\alpha > 0$ such that*

$$\lim_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(E_q^- + E) - \mathcal{N}_b(E_q^-))|}{\ln E} = -\alpha. \tag{2.11}$$

Remarks: i) It is believed that under the rational-flux assumption the Floquet eigenvalues E_j, $j \in \mathbb{N}$, for the operator H^- generically are not constant. Note that this property may hold only generically due to the obvious counterexample where $u = \mathbf{1}_{\Lambda_1}$, $H^- = H_0 + \omega_-$, and for all $j \in \mathbb{N}$ the Floquet eigenvalue E_j is identically equal to $2b(j-1) + \omega_-$. Also, in contrast to the non-magnetic case, we do not know whether the edges in the spectrum of H^- generically are simple.
ii) The definition of the constant α in (2.11) is completely analogous to the one in (2.10) which concerns the non-magnetic case. This definition involving the concepts of Newton polygon, Newton diagram, and Newton decay exponent, is not trivial, and can be found in the original work [25], or in [22, Subsection 4.2.8].

3 Periodic approximation

Pick $a > 0$ such that $\frac{ba^2}{2\pi} \in \mathbb{N}$. Set $L := (2n+1)/2$, $n \in \mathbb{N}$, and define the random $2L\mathbb{Z}^2$-periodic potential

$$V^{\rm per}(x) = V^{\rm per}_{n,\omega}(x) := \sum_{\gamma \in 2L\mathbb{Z}^2} \left(V_\omega \mathbf{1}_{\Lambda_{2L}}\right)(x+\gamma), \quad x \in \mathbb{R}^2.$$

On the domain of H_0 define the operator $H^{\rm per} = H^{\rm per}_{n,\omega} := H_0 + V^{\rm per}_{n,\omega}$. For brevity set $\mathbb{T}_{2L} := \mathbb{T}_{2L\mathbb{Z}^2}$, $\mathbb{T}^*_{2L} := \mathbb{T}^*_{2L\mathbb{Z}^2}$ (see (2.5)). Note that the square Λ_{2L} is the fundamental domain of the torus $\mathbb{T}_{2L}$, while $\Lambda^*_{2L} := \Lambda_{\pi L^{-1}}$ is the fundamental domain of $\mathbb{T}^*_{2L}$. As in (2.7), on the domain of h_0 define the operator

$$h(\theta) = h^{\rm per}(\theta) := h_0(\theta) + V^{\rm per}, \quad \theta \in \mathbb{T}^*_{2L},$$

and by analogy with (2.8) set

$$\mathcal{H}^{\rm per} := \int_{\Lambda^*_{2L}} \oplus\, h^{\rm per}(\theta) d\theta.$$

As above, the operators H_0 and $H^{\rm per}$ are unitarily equivalent to the operators $\mathcal{H}_0$ and $\mathcal{H}^{\rm per}$ respectively. Set

$$\mathcal{N}^{\rm per}(E) = \mathcal{N}^{\rm per}_{n,\omega}(E) := (2\pi)^{-2} \int_{\Lambda^*_{2L}} N(E; h^{\rm per}(\theta))d\theta, \quad E \in \mathbb{R}. \tag{3.1}$$

Here and in the sequel, if T is a self-adjoint operator with purely discrete spectrum, then $N(E;T)$ denotes the number of the eigenvalues of T less than $E \in \mathbb{R}$, and counted with the multiplicities. The function $\mathcal{N}^{\rm per}$ plays the role of IDS for the operator $H^{\rm per}$ since, similarly to (1.4) and (1.5), we have

$$\int_{\mathbb{R}} \varphi(E)d\mathcal{N}^{\rm per}(E) = \lim_{R\to\infty} R^{-2}{\rm Tr}\,\left(\mathbf{1}_{\Lambda_R}\varphi(H^{\rm per})\mathbf{1}_{\Lambda_R}\right)$$

almost surely, and

$$\mathbb{E}\left(\int_{\mathbb{R}} \varphi(E)d\mathcal{N}^{\rm per}(E)\right) = \mathbb{E}\left({\rm Tr}\,\left(\mathbf{1}_{\Lambda_1}\varphi(H^{\rm per})\mathbf{1}_{\Lambda_1}\right)\right), \tag{3.2}$$

for any $\varphi \in C_0^\infty(\mathbb{R})$ (see e.g. the proof of [21, Theorem 5.1] where however the case of a vanishing magnetic field is considered).

Theorem 3.1. *Assume that hypotheses* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *hold. Let* $q \in \mathbb{Z}_+$, $\eta > 0$. *Then there exist* $\nu > 0$ *and* $E_0 > 0$ *such that for* $E \in (0, E_0]$ *and* $n \geq E^{-\nu}$ *we have*

$$\mathbb{E}\left(\mathcal{N}^{\rm per}(2bq + E/2) - \mathcal{N}^{\rm per}(2bq - E/2)\right) - e^{-E^{-\eta}} \leq \mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq - E) \leq$$

$$\mathbb{E}\left(\mathcal{N}^{\rm per}(2bq + 2E) - \mathcal{N}^{\rm per}(2bq - 2E)\right) + e^{-E^{-\eta}}. \tag{3.3}$$

The main technical steps of the proof of Theorem 3.1 which is the central result of this section, are contained in Lemmas 3.1 and 3.2 below.

Lemma 3.1. *Let* $Q = \overline{Q} \in L^\infty(\mathbb{R}^2)$, $X := H_0 + Q$, $D(X) = D(H_0)$. *Then there exists* $\epsilon = \epsilon(b) > 0$ *such that for each* $\alpha, \beta \in \mathbb{Z}^2$, *and* $z \in \mathbb{C} \setminus \sigma(X)$ *we have*

$$\|\chi_\alpha(X - z)^{-1}\chi_\beta\|_{\rm HS} \leq 2\frac{b+1}{\pi^{1/2}}\left(1 + \frac{1}{\eta(z)}\right) e^{-\epsilon\eta(z)|\alpha-\beta|} \tag{3.4}$$

where $\chi_\alpha := \mathbf{1}_{\Lambda_1+\alpha}$, $\alpha \in \mathbb{Z}^2$, $\eta(z) = \eta(z; b, Q) := \frac{{\rm dist}(z,\sigma(X))}{|z|+|Q|_\infty+1}$, $\|\cdot\|_{\rm HS}$ *denotes the Hilbert-Schmidt norm, and* $|Q|_\infty := \|Q\|_{L^\infty(\mathbb{R}^2)}$.

Proof. We will apply the ideas of the proof of [20, Proposition 4.1]. For $\xi \in \mathbb{R}^2$ set

$$X_\xi := e^{\xi\cdot x}Xe^{-\xi\cdot x} = (i\nabla + A - i\xi)^2 + Q = X - 2i\xi\cdot(i\nabla + A) + |\xi|^2.$$

Evidently,

$$X_\xi - z = (X-z)\left(1 + (X-z)^{-1}\left(|\xi|^2 - 2i\xi\cdot(i\nabla + A)\right)\right). \tag{3.5}$$

Let us estimate the norm of the operator $(X-z)^{-1}\left(|\xi|^2 - 2i\xi\cdot(i\nabla+A)\right)$ appearing at the right-hand side of (3.5). We have

$$\|(X-z)^{-1}|\xi|^2\| \le |\xi|^2 \mathrm{dist}(z,\sigma(X))^{-1},$$

$$\|(X-z)^{-1}2i\xi\cdot(i\nabla+A)\| \le$$
$$2\|(H_0+1)^{-1}(i\nabla+A)\cdot\xi - (X-z)^{-1}(Q-z-1)(H_0+1)^{-1}(i\nabla+A)\cdot\xi\| \le$$
$$2C\left(1+\frac{1}{\eta(z)}\right)|\xi|$$

with

$$C = C(b) := \|(H_0+1)^{-1}(i\nabla+A)\| = \sup_{q\in\mathbb{Z}_+}\frac{((2q+1)b)^{1/2}}{2bq+1}.$$

Choose $\epsilon \in \left(0, \frac{1}{8(C+1)}\right)$ and $\xi\in\mathbb{R}^2$ such that $|\xi| = \epsilon\eta(z)$. Then, by the above estimates, we have

$$\|(X-z)^{-1}\left(|\xi|^2 - 2i\xi\cdot(i\nabla+A)\right)\| \le \epsilon^2\eta(z)^2\mathrm{dist}(z,\sigma(X))^{-1} + 2C\epsilon\left(1+\frac{1}{\eta(z)}\right)\eta(z) \le$$

$$\epsilon^2\eta(z) + 2C\epsilon(1+\eta(z)) < \epsilon^2 + 4C\epsilon < 3/4 \tag{3.6}$$

since the resolvent identity implies $\eta(z) < 1$. Therefore, the operator $X_\xi - z$ is invertible, and

$$\chi_\alpha(X-z)^{-1}\chi_\beta = \left(e^{-\xi\cdot x}\chi_\alpha\right)\chi_\alpha(X_\xi - z)^{-1}\chi_\beta\left(e^{\xi\cdot x}\chi_\beta\right). \tag{3.7}$$

Moreover, (3.5) and (3.6) imply

$$\|\chi_\alpha(X_\xi-z)^{-1}\chi_\beta\|_{\mathrm{HS}} \le 4\|(X-z)^{-1}\chi_\beta\|_{\mathrm{HS}} \le$$

$$4\|(H_0+1)^{-1}\chi_\beta - (X-z)^{-1}(Q-z-1)(H_0+1)^{-1}\chi_\beta\|_{\mathrm{HS}} \le$$

$$4\|(H_0+1)^{-1}\chi_\beta\|_{\mathrm{HS}}(1+\|(X-z)^{-1}(Q-z-1)\|) \le 4\|(H_0+1)^{-1}\chi_\beta\|_{\mathrm{HS}}\left(1+\frac{1}{\eta(z)}\right). \tag{3.8}$$

Finally, applying the diamagnetic inequality for Hilbert-Schmidt operators (see e.g. [1]), we get

$$\|(H_0+1)^{-1}\chi_\beta\|_{\mathrm{HS}} \le \|(H_0+1)^{-1}(H_0+b+1)\|\|(H_0+b+1)^{-1}\chi_\beta\|_{\mathrm{HS}} \le$$

$$\|(H_0+1)^{-1}(H_0+b+1)\|\|(-\Delta+1)^{-1}\chi_\beta\|_{\mathrm{HS}} =$$

$$\sup_{q\in\mathbb{Z}_+} \frac{2bq+b+1}{2bq+1} \, \|(-\Delta+1)^{-1}\chi_\beta\|_{\mathrm{HS}} = \frac{b+1}{2\pi^{1/2}}. \tag{3.9}$$

The combination of (3.7), (3.8), and (3.9) yields

$$\|\chi_\alpha(X-z)^{-1}\chi_\beta\|_{\mathrm{HS}} \le \frac{2(b+1)}{\pi^{1/2}} e^{-\xi(\alpha-\beta)} \left(1+\frac{1}{\eta(z)}\right).$$

Choosing $\xi = \epsilon\eta(z)\frac{\alpha-\beta}{|\alpha-\beta|}$, we get (3.4). $\square$

Lemma 3.2. *Assume that hypotheses* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *hold. Then there exists a constant* $C>1$ *such that for any* $\varphi \in C_0^\infty(\mathbb{R})$*, and any* $n\in\mathbb{N}$*,* $l\in\mathbb{N}$*, we have*

$$\left|\mathbb{E}\left(\int_{\mathbb{R}} \varphi(E)d\mathcal{N}_b(E) - \int_{\mathbb{R}} \varphi(E)d\mathcal{N}^{\mathrm{per}}(E)\right)\right| \le$$

$$cn^{-l}e^{Cl\log l} \sup_{x\in\mathbb{R},\, 0\le j\le l+5} \left|(|x|+C)^{l+5}\frac{d^j\varphi}{dx^j}(x)\right|. \tag{3.10}$$

Proof. We will follow the general lines of the proof of [23, Lemma 2.1]. Due to the fact that we consider only the two-dimensional case, and an alloy-type potential which is almost surely bounded, the argument here is somewhat simpler than the one in [23]. By (1.5) and (3.2) we have

$$\mathbb{E}\left(\int_{\mathbb{R}} \varphi(E)d\mathcal{N}_b(E) - \int_{\mathbb{R}} \varphi(E)d\mathcal{N}^{\mathrm{per}}(E)\right) = \mathbb{E}\left(\mathrm{Tr}\,\left(\mathbf{1}_{\Lambda_1}(\varphi(H)-\varphi(H^{\mathrm{per}}))\mathbf{1}_{\Lambda_1}\right)\right).$$

Next, we introduce a representation of the operator $\varphi(H)-\varphi(H^{\mathrm{per}})$ by the Helffer-Sjöstrand formula (see e.g. [4, Chapter 8]). Let $\tilde\varphi$ be an almost analytic extension of the function $\varphi\in C_0^\infty(\mathbb{R})$ appearing in (3.10). We recall that $\tilde\varphi$ possesses the following properties:

1. If $\mathrm{Im}\, z = 0$, then $\tilde\varphi(z)=\varphi(z)$.
2. $\mathrm{supp}\,\tilde\varphi \subset \{x+iy\in\mathbb{C};\, |y|<1\}$.
3. $\tilde\varphi \in \mathcal{S}\left(\{x+iy\in\mathbb{C};\, |y|<1\}\right)$.
4. The family of functions $x\mapsto \frac{\partial\tilde\varphi}{\partial\bar z}(x+iy)|y|^{-m}$, $|y|\in(0,1)$, is bounded in $\mathcal{S}(\mathbb{R})$ for any $m\in\mathbb{Z}_+$.

Such extensions exist for $\varphi\in\mathcal{S}(\mathbb{R})$ (see [27], [4, Chapter 8]), and there exists a constant $C>1$ such that for any $m\ge0$, $\alpha\ge0$, $\beta\ge0$, we have

$$\sup_{0\le|y|\le1}\sup_{x\in\mathbb{R}}\left|x^\alpha\frac{\partial^\beta}{\partial x^\beta}\left(|y|^{-m}\frac{\partial\tilde\varphi}{\partial\bar z}(x+iy)\right)\right| \le$$

$$C^{m\log m+\alpha\log\alpha+\beta+1}\sup_{\beta'\leq m+\beta+2,\ \alpha'\leq\alpha}\sup_{x\in\mathbb{R}}\left|x^{\alpha'}\frac{d^{\beta'}\varphi(x)}{dx^{\beta'}}\right|. \tag{3.11}$$

Then the Helffer-Sjöstrand formula yields

$$\mathbb{E}\left(\mathrm{Tr}\ (\mathbf{1}_{\Lambda_1}(\varphi(H)-\varphi(H^{\mathrm{per}}))\mathbf{1}_{\Lambda_1})\right)=$$

$$\frac{1}{\pi}\mathbb{E}\left(\mathrm{Tr}\ \left(\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\left(\mathbf{1}_{\Lambda_1}\left((H-z)^{-1}-(H^{\mathrm{per}}-z)^{-1}\right)\mathbf{1}_{\Lambda_1}\right)dxdy\right)\right)=$$

$$\frac{1}{\pi}\mathbb{E}\left(\mathrm{Tr}\ \left(\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\left(\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\mathbf{1}_{\Lambda_1}\right)dxdy\right)\right). \tag{3.12}$$

Next, we will show that $\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\mathbf{1}_{\Lambda_1}$ is a trace-class operator for $z\in\mathbb{C}\setminus\mathbb{R}$, and almost surely

$$\|\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\mathbf{1}_{\Lambda_1}\|_{\mathrm{Tr}}\leq\frac{M(b+1)^2}{2\pi}\left(1+\frac{M+|z|+1}{|\mathrm{Im\ z}|}\right)^2 \tag{3.13}$$

where $\|.\|_{\mathrm{Tr}}$ denotes the trace-class norm. Evidently,

$$\|\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\mathbf{1}_{\Lambda_1}\|_{\mathrm{Tr}}\leq$$

$$\|\mathbf{1}_{\Lambda_1}(H_0+1)^{-1}\|^2_{\mathrm{HS}}\|(V^{\mathrm{per}}-V)\|\|(H_0+1)(H-z)^{-1}\|\|(H_0+1)(H^{\mathrm{per}}-z)^{-1}\|. \tag{3.14}$$

By (3.9) we have $\|\mathbf{1}_{\Lambda_1}(H_0+1)^{-1}\|^2_{\mathrm{HS}}\leq\frac{(b+1)^2}{4\pi}$. Moreover, almost surely $\|V^{\mathrm{per}}-V\|\leq 2M$. Finally, it is easy to check that both norms $\|(H_0+1)(H-z)^{-1}\|$ and $\|(H_0+1)(H^{\mathrm{per}}-z)^{-1}\|$ are almost surely bounded from above by $1+\frac{M+|z|+1}{|\mathrm{Im\ z}|}$, so that (3.13) follows from (3.14). Taking into account estimate (3.13) and Properties 2, 3, and 4 of the almost analytic continuation $\tilde{\varphi}$, we find that (3.12) implies

$$\mathbb{E}\left(\mathrm{Tr}\ (\mathbf{1}_{\Lambda_1}(\varphi(H)-\varphi(H^{\mathrm{per}}))\mathbf{1}_{\Lambda_1})\right)=$$

$$\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\mathbf{1}_{\Lambda_1}\right)\right)dxdy. \tag{3.15}$$

Our next goal is to obtain a precise estimate (see (3.19) below) on the decay rate as $n\to\infty$ of

$$\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\mathbf{1}_{\Lambda_1}\right)\right)$$

with $z\in\mathbb{C}\setminus\mathbb{R}$ and $|\mathrm{Im}\ z|<1$. Evidently,

$$\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\mathbf{1}_{\Lambda_1}\right)\right)=$$

$$\sum_{\alpha\in\mathbb{Z}^2,|\alpha|_\infty>na}\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}\left((H-z)^{-1}\chi_\alpha(V^{\mathrm{per}}-V)(H^{\mathrm{per}}-z)^{-1}\right)\mathbf{1}_{\Lambda_1}\right)\right)$$

where $|\alpha|_\infty := \max_{j=1,2}|\alpha_j|$, since $V^{\rm per} = V$ on Λ_{2L}, and therefore $\chi_\alpha(V^{\rm per} - V) = 0$ if $|\alpha|_\infty \leq na$. Hence, bearing in mind estimates (1.3) and (3.4), we easily find that

$$|\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\rm per}-V)(H^{\rm per}-z)^{-1}\mathbf{1}_{\Lambda_1}\right)\right)| \leq$$

$$\sum_{\alpha\in\mathbb{Z}^2, |\alpha|_\infty > na} \mathbb{E}\left(\|\chi_0(H-z)^{-1}\chi_\alpha(V^{\rm per}-V)(H^{\rm per}-z)^{-1}\chi_0\|_{\rm Tr}\right) \leq$$

$$2M \sum_{\alpha\in\mathbb{Z}^2, |\alpha|_\infty > na} \mathbb{E}\left(\|\chi_0(H-z)^{-1}\chi_\alpha\|_{\rm HS}\|\chi_\alpha(H^{\rm per}-z)^{-1}\chi_0\|_{\rm HS}\right) \leq$$

$$\frac{M(b+1)^2}{2\pi}\left(1+\frac{|x|+M+2}{|y|}\right)^2 \sum_{\alpha\in\mathbb{Z}^2, |\alpha|_\infty > na} \exp\left(-\frac{2\epsilon|\alpha||y|}{|x|+M+2}\right) \tag{3.16}$$

for every $z = x+iy$ with $0 < |y| < 1$. Using the summation formula for a geometric series, and some elementary estimates, we conclude that there exists a constant C depending only on ϵ such that

$$\sum_{\alpha\in\mathbb{Z}^2, |\alpha|_\infty > na} \exp\left(-\frac{2\epsilon|\alpha||y|}{|x|+M+2}\right) \leq \left(1+C\frac{|x|+M+2}{|y|}\right)\exp\left(-\frac{a\epsilon n|y|}{|x|+M+2}\right) \tag{3.17}$$

provided that $0 < |y| < 1$. Putting together (3.16) and (3.17), we find that there exists a constant $C = C(M, b, \epsilon, a)$ such that

$$\left|\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\rm per}-V)(H^{\rm per}-z)^{-1}\mathbf{1}_{\Lambda_1}\right)\right)\right| \leq C\left(\frac{|x|+C}{|y|}\right)^3 \exp\left(-\frac{a\epsilon n|y|}{|x|+C}\right). \tag{3.18}$$

Writing

$$\left(\frac{|x|+C}{|y|}\right)^3 \exp\left(-\frac{a\epsilon n|y|}{|x|+C}\right) = (a\epsilon n)^{-l}\left(\frac{|x|+C}{|y|}\right)^{3+l}\left(\frac{a\epsilon n|y|}{|x|+C}\right)^l \exp\left(-\frac{a\epsilon n|y|}{|x|+C}\right)$$

with $l \in \mathbb{N}$, and bearing in mind the elementary inequality $t^l e^{-t} \leq (l/e)^l$, $t \geq 0$, $l \in \mathbb{N}$, we find that (3.18) implies

$$\left|\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}(H-z)^{-1}(V^{\rm per}-V)(H^{\rm per}-z)^{-1}\mathbf{1}_{\Lambda_1}\right)\right)\right| \leq$$

$$C(a\epsilon e)^{-l} n^{-l}\left(\frac{|x|+C}{|y|}\right)^{3+l} e^{l\log l}, \quad l \in \mathbb{N}. \tag{3.19}$$

Combining (3.19) and (3.15), we get

$$|\mathbb{E}\left(\mathrm{Tr}\ \left(\mathbf{1}_{\Lambda_1}(\varphi(H)-\varphi(H^{\rm per}))\mathbf{1}_{\Lambda_1}\right)\right)| \leq$$

$$\frac{C}{\pi}\int_{\mathbb{R}}(|x|+C)^{-2}dx\,(a\epsilon e)^{-l}n^{-l}e^{l\log l}\sup_{0<|y|<1}\sup_{x\in\mathbb{R}}(|x|+C)^{l+5}|y|^{-(l+3)}\left|\frac{\partial\tilde{\varphi}}{\partial\bar{z}}(x+iy)\right|,\quad l\in\mathbb{N}. \tag{3.20}$$

Applying estimate (3.11) on almost analytic extensions, we find that (3.20) entails (3.10). □

Now we are in position to prove Theorem 3.1. Let $\varphi_+ \in C_0^\infty(\mathbb{R})$ be a non-negative Gevrey-class function with Gevrey exponent $\varrho > 1$, such that $\int_{\mathbb{R}}\varphi_+(t)dt = 1$, $\operatorname{supp}\varphi_+ \subset \left[-\frac{E}{2}, \frac{E}{2}\right]$. Set $\Phi_+ := \mathbf{1}_{\left[2bq-\frac{3E}{2}, 2bq+\frac{3E}{2}\right]} * \varphi_+$. Then Φ_+ is Gevrey-class function with Gevrey exponent ϱ. Moreover,

$$\mathbf{1}_{[2bq-E,2bq+E]}(t) \le \Phi_+(t) \le \mathbf{1}_{[2bq-2E,2bq+2E]}(t),\quad t\in\mathbb{R}.$$

Therefore,

$$\mathcal{N}_b(2bq+E)-\mathcal{N}_b(2bq-E) \le \mathbb{E}\left(\mathcal{N}^{\rm per}(2bq+2E)-\mathcal{N}^{\rm per}(2bq-2E)\right) +$$

$$\left|\mathbb{E}\left(\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}_b(t)-\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}^{\rm per}(t)\right)\right|. \tag{3.21}$$

Applying Lemma 3.2 and the standard estimates on the derivatives of Gevrey-class functions, we get

$$\left|\mathbb{E}\left(\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}_b(t)-\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}^{\rm per}(t)\right)\right| \le Cn^{-l}(l+5)^{\varrho(l+5)},\quad l\in\mathbb{N}, \tag{3.22}$$

with C independent of n, and l. Optimizing the r.h.s. of (3.22) with respect to l, we get

$$\left|\mathbb{E}\left(\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}_b(t)-\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}^{\rm per}(t)\right)\right| \le \exp\left(-(\varrho+C)n^{1/(\varrho+C)}\right)$$

for sufficiently large n. Picking $\eta > 0$, and choosing $\nu > (\varrho+C)\eta$ and $n \ge E^{-\nu}$, we find that

$$\left|\mathbb{E}\left(\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}_b(t)-\int_{\mathbb{R}}\Phi_+(t)d\mathcal{N}^{\rm per}(t)\right)\right| \le e^{-E^{-\eta}} \tag{3.23}$$

for sufficiently small $E > 0$. Now the combination of (3.21) and (3.23) yields the upper bound in (3.3). The proof of the first inequality in (3.3) is quite similar, so that we will just outline it. Let $\varphi_- \in C_0^\infty(\mathbb{R})$ be a non-negative Gevrey-class function with Gevrey exponent $\varrho > 1$, such that $\int_{\mathbb{R}}\varphi_+(t)dt = 1$, and $\operatorname{supp}\varphi_+ \subset \left[-\frac{E}{4}, \frac{E}{4}\right]$. Set $\Phi_+ := \mathbf{1}_{\left[2bq-\frac{3E}{4}, 2bq+\frac{3E}{4}\right]} * \varphi_+$. Then Φ_- is Gevrey-class function with Gevrey exponent ϱ. Similarly to (3.21) we have

$$\mathbb{E}\left(\mathcal{N}^{\rm per}(2bq+E/2)-\mathcal{N}^{\rm per}(2bq-E/2)\right) - \left|\int_{\mathbb{R}}\mathbb{E}\left(\Phi_-(t)d\mathcal{N}_b(t)-\int_{\mathbb{R}}\Phi_-(t)d\mathcal{N}^{\rm per}(t)\right)\right| \le$$

$$\leq \mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq - E). \tag{3.24}$$

Arguing as in the proof of (3.23), we obtain

$$\left| \int_{\mathbb{R}} \mathbb{E} \left(\Phi_-(t) d\mathcal{N}_b(t) - \int_{\mathbb{R}} \Phi_-(t) d\mathcal{N}^{\text{per}}(t) \right) \right| \leq e^{-E^{-\eta}}$$

which combined with (3.24) yields the lower bound in (3.3). Thus, the proof of Theorem 3.1 is now complete.

Further, we introduce a reduced IDS ρ_q related to a fixed Landau level $2bq$, $q \in \mathbb{Z}_-$.

It is well-known that for every fixed $\theta \in \mathbb{T}^*_{2L}$ we have $\sigma(h(\theta)) = \cup_{q=0}^{\infty} \{2bq\}$, and $\dim \text{Ker } (h(\theta) - 2bq) = 2bL^2/\pi$ for each $q \in \mathbb{Z}_+$ (see [5]). Denote by $p_q(\theta) : L^2(\Lambda_{2L}) \to L^2(\Lambda_{2L})$ the orthogonal projection onto Ker $(h(\theta) - 2bq)$, and by $r_q(\theta) = r_{q,n,\omega}(\theta)$ the operator $p_q(\theta) V^{\text{per}}_{n,\omega} p_q(\theta)$ defined and self-adjoint on the finite-dimensional Hilbert space $p_q(\theta) L^2(\Lambda_{2L})$. Set

$$\rho_q(E) = \rho_{q,n,\omega}(E) = (2\pi)^{-2} \int_{\Lambda^*_{2L}} N(E; r_{q,n,\omega}(\theta)) d\theta, \quad E \in \mathbb{R}. \tag{3.25}$$

By analogy with (3.1), we call the function $\rho_{q,n,\omega}$ the IDS for the operator $\mathcal{R}_q = \mathcal{R}_{q,n,\omega} := \int_{\Lambda^*_{2L}} \oplus r_{q,n,\omega} d\theta$ defined and self-adjoint on $\mathcal{P}_q L^2(\Lambda_{2L} \times \Lambda^*_{2L})$ where $\mathcal{P}_q := \int_{\Lambda^*_{2L}} \oplus p_q(\theta) d\theta$. Note that $\mathcal{R}_q = \mathcal{P}_q V^{\text{per}} \mathcal{P}_q$.

Denote by P_q, $q \in \mathbb{Z}_+$, the orthogonal projection onto $\text{Ker}(H_0 - 2bq)$. Evidently, $\mathcal{P}_q = U P_q U^*$. As mentioned in the Introduction, rank $P_q = \infty$ for every $q \in \mathbb{Z}_+$. Moreover, the functions

$$e_j(x) = e_{j,q}(x) := (-i)^q \sqrt{\frac{q!}{\pi j!}} \left(\frac{b}{2}\right)^{(j-q+1)/2} (x_1 + ix_2)^{j-q} L_q^{(j-q)} \left(\frac{b}{2}|x|^2\right) e^{-\frac{b}{4}|x|^2}, \; j \in \mathbb{Z}_+, \tag{3.26}$$

form the so-called angular-momentum orthogonal basis of $P_q L^2(\mathbb{R}^2)$, $q \in \mathbb{Z}_+$ (see [8] or [3, Section 9]). Here

$$L_q^{(j-q)}(\xi) := \sum_{l=\max\{0,q-j\}}^{q} \frac{j!}{(j-q+l)!(q-l)!} \frac{(-\xi)^l}{l!}, \quad \xi \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \quad j \in \mathbb{Z}_+,$$

are the generalized Laguerre polynomials. For further references we give here several estimates concerning the functions $e_{j,k}$. If $q \in \mathbb{Z}_+$, $j \geq 1$, and $\xi \geq 0$, we have

$$L_q^{(j-q)}(j\xi)^2 \leq j^{2q} e^{2\xi} \tag{3.27}$$

(see [14, Eq. (4.2)]). On the other hand, there exists $j_0 > q$ such that $j \geq j_0$ implies

$$L_q^{(j-q)}(j\xi)^2 \geq \frac{1}{(q!)^2} \left(\frac{1}{2}\right)^{2+2q} (j-q)^{2q} \tag{3.28}$$

if $\xi \in [0,1/2]$ (see [32, Eq. (3.6)]). Moreover, for $j \in \mathbb{Z}_+$ and $q \in \mathbb{Z}_+$ we have

$$e_{j,q}(x) = \frac{1}{\sqrt{q!(2b)^q}}(a^*)^q e_{0,q}(x), \quad x \in \mathbb{R}, \tag{3.29}$$

where

$$a^* := -i\frac{\partial}{\partial x_1} - A_1 - i\left(-i\frac{\partial}{\partial x_2} - A_2\right) = -2ie^{b|z|^2/4}\frac{\partial}{\partial z}e^{-b|z|^2/4}, \quad z := x_1 + ix_2, \tag{3.30}$$

is the creation operator (see e.g. [3, Section 9]). Evidently, a^* commutes with the magnetic translation operators τ_γ, $\gamma \in 2L\mathbb{Z}^2$ (see (2.6)). Finally, the projection P_q, $q \in \mathbb{Z}_+$, admits the integral kernel

$$K_{q,b}(x,x') = \frac{b}{2\pi}e^{-i\frac{b}{2}x\wedge x'}\Psi_q\left(\frac{b}{2}|x-x'|^2\right), \quad x, x' \in \mathbb{R}^2, \tag{3.31}$$

where $\Psi_q(\xi) := L_q^{(0)}(\xi)e^{-\xi/2}$, $\xi \in \mathbb{R}$. Since P_q is an orthogonal projection in $L^2(\mathbb{R}^2)$ we have $\|P_q\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} = 1$. Using the facts that $\mathcal{P}_q = UP_qU^*$ and $\mathcal{P}_q := \int_{\Lambda_{2L}^*} \oplus p_q(\theta)d\theta$, as well as the explicit expressions (2.9) for the unitary operator U, and (3.31) for the integral kernel of P_q, $q \in \mathbb{Z}_+$, we easily find that the projection $p_q(\theta)$, $\theta \in \mathbb{T}_{2L}^*$, admits an explicit kernel in the form

$$\mathcal{K}_{q,b}(x,x';\theta) = \frac{b}{2\pi}e^{i\theta(x'-x)}e^{-i\frac{b}{2}x\wedge x'}\times$$

$$\sum_{\alpha\in 2L\mathbb{Z}^2}\Psi_q\left(\frac{b}{2}|x-x'+\alpha|^2\right)e^{-i\theta\alpha}e^{i\frac{b}{2}(x+x')\wedge\alpha}e^{i\frac{b}{2}\alpha_1\alpha_2}, \quad x, x' \in \Lambda_{2L}. \tag{3.32}$$

Lemma 3.3. *Let the assumptions of Theorem 3.1 hold. Suppose, moreover, that the random variables ω_γ, $\gamma \in \mathbb{Z}^2$, are non-negative.*
a) For each $c_0 \in \left(1 + \frac{M}{2b}, \infty\right)$ there exists $E_0 \in (0, 2b)$ such that for each $E \in (0, E_0)$, $\theta \in \mathbb{T}_{2L}^*$, almost surely

$$N(E; r_0(\theta)) \le N(E; h(\theta)) \le N(c_0 E; r_0(\theta)). \tag{3.33}$$

b) Assume $\mathbf{H}_4$, i.e. $2b > M$. Then for each $c_1 \in \left(0, 1 - \frac{M}{2b}\right)$, $c_2 \in \left(1 + \frac{M}{2b}, \infty\right)$, there exists $E_0 \in (0, 2b)$ such that for each $E \in (0, E_0)$, $\theta \in \mathbb{T}_{2L}^*$, and $q \ge 1$, almost surely

$$N(c_1 E; r_q(\theta)) \le N(2bq + E; h(\theta)) - N(2bq; h(\theta)) \le N(c_2 E; r_q(\theta)). \tag{3.34}$$

Proof. In order to simplify the notations we will omit the explicit dependence of the operators h, h_0, p_q, and r_q, on $\theta \in \mathbb{T}_{2L}^*$. Moreover, we set $\mathcal{D}_q := p_q D(h) = p_q L^2(\Lambda_{2L})$, and $\mathcal{C}_q := (1 - p_q)D(h)$. At first we prove (3.33). The minimax principle implies

$$N(E; h) \ge N(E; p_0 h p_{0|\mathcal{D}_0}) = N(E; r_0)$$

which coincides with the lower bound in (3.33). On the other hand, the operator inequality

$$h \geq p_0(h_0 + (1-\delta)V^{\rm per})p_0 + (1-p_0)(h_0 + (1-\delta^{-1})V^{\rm per})(1-p_0), \quad \delta \in (0,1), \tag{3.35}$$

combined with the minimax principle, entails

$$\begin{aligned} N(E;h) &\leq N(E; p_0(h_0 + (1-\delta)V^{\rm per})p_{0|\mathcal{D}_0}) \\ &\qquad\qquad + N(E;(1-p_0)(h_0 + (1-\delta^{-1})V^{\rm per})(1-p_0)_{|\mathcal{C}_0}) \\ &\leq N((1-\delta)^{-1}E; r_0) + N(E + M(\delta^{-1}-1); (1-p_0)h_0(1-p_0)_{|\mathcal{C}_0}). \end{aligned} \tag{3.36}$$

Choose $M(\delta^{-1}-1) < 2b$, and, hence, $c_0 := (1-\delta)^{-1} > 1 + \frac{M}{2b}$, and $E \in (0, 2b - M(\delta^{-1} - 1))$. Since

$$\inf \sigma((1-p_0)h_0(1-p_0)_{|\mathcal{C}_0}) = 2b,$$

we find that the second term on the r.h.s. of (3.36) vanishes, and $N(E;h) \leq N(c_0E;r_0)$ which coincides with the upper bound in (3.33).
Next we assume $q \geq 1$ and $M < 2b$, and prove (3.34). Note for any $E_1 \in (0, 2b - M)$ we have

$$N(2bq;h) = N(2bq - E_1; h).$$

Pick again $\delta \in \left(\frac{M}{2b+M}, 0\right)$ so that $c_2 := (1-\delta)^{-1} > 1 + \frac{M}{2b}$. Then the operator inequality

$$h \geq p_q(h_0 + (1-\delta)V^{\rm per})p_q + (1-p_q)(h_0 + (1-\delta^{-1})V^{\rm per})(1-p_q), \quad \delta \in (0,1),$$

analogous to (3.35), yields

$$\begin{aligned} N(2bq+E;h) &\leq N(2bq+E; p_q(h_0 + (1-\delta)V^{\rm per})p_{q|\mathcal{D}_q}) \\ &\qquad\qquad + N(2bq+E;(1-p_q)(h_0 + (1-\delta^{-1})V^{\rm per})(1-p_q)_{|\mathcal{C}_q}) \\ &\leq N(c_2E; r_q) + N(2bq + E + M(\delta^{-1}-1); (1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}). \end{aligned}$$

On the other hand, the minimax principle implies

$$N(2bq-E_1;h) \geq N(2bq-E_1;(1-p_q)h(1-p_q)_{|\mathcal{C}_q}) \geq N(2bq-E_1-M;(1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}).$$

Thus we get

$$\begin{aligned} N(2bq+E;h) - N(2bq-E_1;h) &\leq N(c_2E;r_q) \\ &\qquad + N(2bq+E+M(\delta^{-1}-1);(1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}) \\ &\qquad - N(2bq-E_1-M;(1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}). \end{aligned} \tag{3.37}$$

It is easy to check that

$$2bq - E_1 - M > 2b(q-1), \quad 2bq + E + M(\delta^{-1}-1) < 2(q+1)b$$

provided that $E \in (0, 2b - M(\delta^{-1} - 1))$. Since

$$\sigma((1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}) \cap (2(q-1)b, 2(q+1)b) = \emptyset,$$

we find that the the r.h.s. of (3.37) is equal to $N(c_2E; r_q)$, thus getting the upper bound in (3.34).

Finally, we prove the lower bound in (3.34). Pick $\zeta \in \left(\frac{M}{2b-M}, \infty\right)$, and, hence $c_1 := (1+\zeta)^{-1} \in \left(0, \frac{M}{2b}\right)$. Bearing in mind the operator inequality

$$h \leq p_q(h_0 + (1+\zeta)V^{\text{per}})p_q + (1-p_q)(h_0 + (1+\zeta^{-1})V^{\text{per}})(1-p_q),$$

and applying the minimax principle, we obtain

$$\begin{aligned} N(2bq+E; h) \geq\ & N(2bq+E; p_q(h_0 + (1+\zeta)V^{\text{per}})p_{q|\mathcal{D}_q}) \\ & + N(2bq+E; (1-p_q)(h_0+(1+\zeta^{-1})V^{\text{per}})(1-p_q)_{|\mathcal{C}_q}) \\ \geq\ & N(c_1E; r_q) + N(2bq+E-M(\zeta^{-1}+1); (1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}). \end{aligned}$$

On the other hand, since $V^{\text{per}} \geq 0$, the minimax principle directly implies

$$N(2bq - E_1; h) \leq N(2bq - E_1; h_0) = N(2bq - E_1; (1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}).$$

Combining the above estimates, we get

$$\begin{aligned} N(2bq+E; h) - N(2bq-E_1; h) \geq\ & N(c_1E; r_q) \\ & - \big|N(2bq+E-M(\zeta^{-1}+1); (1-p_q)h_0(1-p_q)_{|\mathcal{C}_q}) \\ & -N(2bq-E_1; (1-p_q)h_0(1-p_q)_{|\mathcal{C}_q})\big|. \end{aligned} \tag{3.38}$$

Since

$$2(q-1)b < 2bq + E - M(\zeta^{-1}+1) < 2(q+1)b, \quad 2(q-1)b < 2bq - E_1 < 2(q+1)b,$$

provided that $E \in (0, 2b + M(\zeta^{-1}+1))$, we find that the r.h.s of (3.38) is equal to $N(c_1E; r_q)$ which entails the lower bound in (3.34). $\square$

Integrating (3.33) and (3.34) with respect to θ and ω, and combining the results with (3.3), we obtain the following

Corollary 3.1. *Assume that the hypotheses of Theorem 3.1 hold. Let $q \in \mathbb{Z}_+$ $\eta > 0$. If $q \geq 1$, assume $M < 2b$. Then there exist $\nu = \nu(\eta) > 0$, $d_1 \in (0,1)$, $d_2 \in (1, \infty)$, and $\tilde{E}_0 > 0$, such that for each $E \in (0, \tilde{E}_0)$ and $n \geq E^{-\nu}$, we have*

$$\mathbb{E}\left(\rho_{q,n,\omega}(d_1E)\right) - e^{-E^{-\eta}} \leq \mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq) \leq \mathbb{E}\left(\rho_{q,n,\omega}(d_2E)\right) + e^{-E^{-\eta}}. \tag{3.39}$$

4 Proof of Theorem 2.1: upper bounds of the IDS

In this section we obtain the upper bounds of $\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq)$ necessary for the proof of Theorem 2.1.

Theorem 4.1. *Assume that* $\mathbf{H}_1$ – $\mathbf{H}_4$ *hold, that almost surely* $\omega_\gamma \geq 0$, $\gamma \in \mathbb{Z}^2$, *and* (2.1) *is valid. Fix the Landau level* $2bq$, $q \in \mathbb{Z}_+$.
i) Assume that $u(x) \geq C(1+|x|)^{-\varkappa}$, $x \in \mathbb{R}^2$, *for some* $\varkappa > 2$, *and* $C > 0$. *Then we have*

$$\liminf_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq))|}{|\ln E|} \geq \frac{2}{\varkappa - 2}. \tag{4.1}$$

ii) Assume $u(x) \geq Ce^{-C|x|^\beta}$, $x \in \mathbb{R}^2$, *for some* $\beta > 0$, $C > 0$. *Then we have*

$$\liminf_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} \geq 1 + \frac{2}{\beta}. \tag{4.2}$$

iii) Assume $u(x) \geq C\mathbf{1}_{\{x \in \mathbb{R}^2\,;\,|x-x_0|<\varepsilon\}}$ *for some* $C > 0$, $x_0 \in \mathbb{R}^2$, *and* $\varepsilon > 0$. *Then there exists* $\delta > 0$ *such that we have*

$$\liminf_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq)|}{\ln |\ln E|} \geq 1 + \delta. \tag{4.3}$$

Fix $\theta \in \mathbb{T}^*_{2L}$. Denote by $\lambda_j(\theta)$, $j = 1, \ldots, \operatorname{rank} r_{q,n,\omega}(\theta)$, the eigenvalues of the operator $r_{q,n,\omega}(\theta)$ enumerated in non-decreasing order. Then (3.25) implies

$$\mathbb{E}\left(\rho_{q,n,\omega}(E)\right) = \frac{1}{(2\pi)^2} \int_{\Lambda^*_{2L}} \mathbb{E}(N(E; r_{q,n,\omega}(\theta)) d\theta = \frac{1}{(2\pi)^2} \int_{\Lambda^*_{2L}} \sum_{j=1}^{\operatorname{rank} r_{q,n,\omega}(\theta)} \mathbb{P}(\lambda_j(\theta) < E) d\theta \tag{4.4}$$

with $E \in \mathbb{R}$. Since the potential V is almost surely bounded, we have $\operatorname{rank} r_{q,n,\omega}(\theta) \leq \operatorname{rank} p_q(\theta) = 2bL^2/\pi$. Therefore, (4.4) entails

$$\mathbb{E}\left(\rho_{q,n,\omega}(E)\right) \leq \frac{bL^2}{2\pi^3} \int_{\Lambda^*_{2L}} \mathbb{P}(r_{q,n,\omega}(\theta) \text{ has an eigenvalue less than } E) d\theta. \tag{4.5}$$

In order to estimate the probability in (4.5), we need the following

Lemma 4.1. *Assume that, for* $n \sim E^{-\nu}$, *the operator* $r_{q,n,\omega}(\theta)$ *has an eigenvalue less than* E. *Set* $L := (2n+1)a/2$. *Pick* E *small and* l *large such that* $L >> l$ *both large. Decompose* $\Lambda_{2L} = \cup_{\gamma \in 2l\mathbb{Z}^2 \cap \Lambda_{2L}} (\gamma + \Lambda_{2l})$. *Fix* $\mathcal{C} > 1$ *sufficiently large and* $m = m(L,l)$ *such that*

$$\frac{1}{\mathcal{C}} bl^2 \leq m \leq \mathcal{C} bL^2, \tag{4.6}$$

$$E\left(\frac{l}{L}\right)^2 > \mathcal{C} e^{-bl^2/2 + m \ln(\mathcal{C} bl^2/m)}. \tag{4.7}$$

Then, there exists $\gamma \in 2l\mathbb{Z}^2 \cap \Lambda_{2L}$ and a non identically vanishing function $\psi \in L^2(\mathbb{R}^2)$ in the span of $\{e_{j,q}\}_{0\leq j\leq m}$, the functions $e_{j,q}$ being defined in (3.26), such that

$$\langle V_\omega^\gamma \psi, \psi\rangle_l \leq 2E\langle \psi, \psi\rangle_l \tag{4.8}$$

where $V_\omega^\gamma(x) = V_\omega^{\mathrm{per}}(x+\gamma)$, and $\langle \cdot, \cdot\rangle_l := \int_{\Lambda_{2l}} |\cdot|^2 dx$.

Proof. Consider $\varphi \in \operatorname{Ran} p_q(\theta)$ a normalized eigenfunction of the operator $r_{q,n,\omega}(\theta)$ corresponding to an eigenvalue smaller than E. Then we have

$$\langle V_\omega \varphi, \varphi\rangle_L \leq E\langle \varphi, \varphi\rangle_L. \tag{4.9}$$

Whenever necessary, we extend φ by magnetic periodicity (i.e. the periodicity with respect to the magnetic translations) to the whole plane $\mathbb{R}^2$. Note that

$$\varphi(x) = \varphi(x;\theta) = \int_{\Lambda_{2L}} \mathcal{K}_{q,b}(x,x';\theta)\varphi(x')dx' = \frac{b}{2\pi}\int_{\mathbb{R}^2} e^{i\theta(x'-x)}K_{q,b}(x,x')\varphi(x')dx'$$

with $x \in \Lambda_{2L}$ (see (3.31) and (3.32) for the definition of $K_{q,b}$ and $\mathcal{K}_{q,b}$ respectively). Evidently, $\varphi \in L^\infty(\mathbb{R}^2)$, and since it is normalized in $L^2(\Lambda_{2L})$, we have

$$\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq \sup_{x\in\Lambda_{2L}} \left(\int_{\Lambda_{2L}} |\mathcal{K}_{q,b}(x,x';\theta)|^2 dx'\right)^{1/2} \leq$$

$$\sup_{x\in\Lambda_{2L}} \left(\int_{\Lambda_{2L}} \left(\sum_{\alpha\in 2L\mathbb{Z}^2} \tilde{\Psi}_q(x-x'+\alpha)\right)^2 dx'\right)^{1/2} \leq C \tag{4.10}$$

where

$$\tilde{\Psi}_q(y) := \frac{b}{2\pi}\left|\Psi_q\left(\frac{b}{2}|y|^2\right)\right|, \quad y \in \mathbb{R}^2. \tag{4.11}$$

and C depends on q and b but is independent of n and θ.
Fix $C_1 > 1$ large to be chosen later on. Consider the sets

$$\mathcal{L}_+ = \left\{\gamma \in 2l\mathbb{Z}^2 \cap \Lambda_{2L};\ \int_{\gamma+\Lambda_{2l}} |\varphi(x)|^2 dx \geq \frac{1}{C_1}\left(\frac{l}{L}\right)^2 \int_{\Lambda_{2L}} |\varphi(x)|^2 dx\right\},$$
$$\mathcal{L}_- = \left\{\gamma \in 2l\mathbb{Z}^2 \cap \Lambda_{2L};\ \int_{\gamma+\Lambda_{2l}} |\varphi(x)|^2 dx < \frac{1}{C_1}\left(\frac{l}{L}\right)^2 \int_{\Lambda_{2L}} |\varphi(x)|^2 dx\right\}.$$

The sets $\mathcal{L}_-$ and $\mathcal{L}_+$ partition $2l\mathbb{Z}^2 \cap \Lambda_{2L}$.
Fix $C_2 > 1$ large. Let us now prove that for some $\gamma \in \mathcal{L}_+$, one has

$$\int_{\gamma+\Lambda_{2l}} V_\omega^{\mathrm{per}}(x)|\varphi(x)|^2 dx \leq C_2 E \int_{\gamma+\Lambda_{2l}} |\varphi(x)|^2 dx. \tag{4.12}$$

Indeed, if this were not the case, then (4.9) would yield

$$\begin{aligned}
-E\sum_{\gamma\in\mathcal{L}_-}\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx &\le \sum_{\gamma\in\mathcal{L}_-}\left(\int_{\gamma+\Lambda_{2l}}V_\omega^{\mathrm{per}}(x)|\varphi(x)|^2dx - E\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx\right)\\
&\le \sum_{\gamma\in\mathcal{L}_+}\left(E\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx - \int_{\gamma+\Lambda_{2l}}V_\omega^{\mathrm{per}}(x)|\varphi(x)|^2dx\right)\\
&\le -E(C_2-1)\sum_{\gamma\in\mathcal{L}_+}\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx.
\end{aligned} \tag{4.13}$$

On the other hand, the definition of $\mathcal{L}_-$ yields

$$\begin{aligned}
\int_{\Lambda_{2L}}|\varphi(x)|^2dx &= \sum_{\gamma\in\mathcal{L}_-}\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx + \sum_{\gamma\in\mathcal{L}_+}\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx\\
&\le \sum_{\gamma\in\mathcal{L}_+}\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx + \frac{1}{C_1}\sum_{\gamma\in\mathcal{L}_-}\left(\frac{l}{L}\right)^2\int_{\Lambda_{2L}}|\varphi(x)|^2dx\\
&\le \sum_{\gamma\in\mathcal{L}_+}\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx + \frac{1}{C_1}\int_{\Lambda_{2L}}|\varphi(x)|^2dx.
\end{aligned}$$

Plugging this into (4.13), we get

$$\frac{E}{C_1}\int_{\Lambda_{2L}}|\varphi(x)|^2dx \ge E(C_2-1)\left(1-\frac{1}{C_1}\right)\int_{\Lambda_{2L}}|\varphi(x)|^2dx \tag{4.14}$$

which is clearly impossible if we choose $(C_2-1)(C_1-1)>1$.
So from now on we assume that $(C_2-1)(C_1-1)>1$. Hence, we can find $\gamma\in 2l\mathbb{Z}^2\cap\Lambda_{2L}$ such that one has

$$\int_{\gamma+\Lambda_{2l}}V_\omega^{\mathrm{per}}(x)|\varphi(x)|^2dx \le C_2E\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx,$$

$$\int_{\gamma+\Lambda_{2l}}|\varphi(x)|^2dx \ge \frac{1}{C_1}\left(\frac{l}{L}\right)^2\int_{\Lambda_{2L}}|\varphi(x)|^2dx.$$

Shifting the variables in the integrals above by γ, we may assume $\gamma=0$ if we replace V_ω^{per} by V_ω^γ. Thus we get

$$\int_{\Lambda_{2l}}V_\omega^\gamma(x)|\varphi(x)|^2dx \le C_2E\int_{\Lambda_{2l}}|\varphi(x)|^2dx,$$

$$\int_{\Lambda_{2l}}|\varphi(x)|^2dx \ge \frac{1}{C_1}\left(\frac{l}{L}\right)^2\int_{\gamma+\Lambda_{2L}}|\varphi(x)|^2dx.$$

Due to the magnetic periodicity of φ, we have

$$\int_{\gamma+\Lambda_{2L}} |\varphi(x)|^2 dx = \int_{\Lambda_{2L}} |\varphi(x)|^2 dx$$

which yields

$$\int_{\Lambda_{2l}} V_\omega(x)|\varphi(x)|^2 dx \leq C_2 E \int_{\Lambda_{2l}} |\varphi(x)|^2 dx, \tag{4.15}$$

$$\int_{\Lambda_{2l}} |\varphi(x)|^2 dx \geq \frac{1}{C_1}\left(\frac{l}{L}\right)^2 \int_{\Lambda_{2L}} |\varphi(x)|^2 dx. \tag{4.16}$$

Let us now show that roughly the same estimates hold true for φ replaced by a function $\psi \in P_q L^2(\mathbb{R}^2)$. Set $\psi := P_q \chi_- e_\theta \varphi$ where $e_\theta(x) := e^{i\theta x}$, $x \in \mathbb{R}^2$, and χ_- denotes the characteristic function of the set $\{x \in \mathbb{R}^2; |x|_\infty < L\}$. Note that $\varphi - \overline{e_\theta}\psi = \overline{e_\theta} P_q \chi_+ e_\theta \varphi$ where χ_+ is the characteristic function of the set $\{x \in \mathbb{R}^2; |x|_\infty \geq L\}$. Let us estimate the $L^2(\Lambda_{2L})$-norm of the function $\varphi - \overline{e_\theta}\psi$. We have

$$\begin{aligned} \|\varphi - \overline{e_\theta}\psi\|_L^2 &:= \|\varphi - \overline{e_\theta}\psi\|_{L^2(\Lambda_{2L})}^2 \\ &= \int_{\Lambda_{2L}} \left| \int_{\mathbb{R}^2} K_{q,b}(x,x')\chi_+(x') e^{i\theta x} \varphi(x') dx' \right|^2 dx \\ &\leq \sup_{x' \in \mathbb{R}^2} |\varphi(x')|^2 \int_{\Lambda_{2L}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\Psi}_q(x-x')\tilde{\Psi}_q(x-x'')\chi_+(x')\chi_+(x'') dx' dx'' dx, \end{aligned} \tag{4.17}$$

the function $\tilde{\Psi}$ being defined in (4.11). Bearing in mind estimate (4.10), and taking into account the Gaussian decay of $\tilde{\Psi}$ at infinity, we easily find that (4.17) implies the existence of a constant $C > 0$ such that for sufficiently large L we have

$$\|\varphi - \overline{e_\theta}\psi\|_L^2 \leq e^{-L/C}.$$

As φ is normalized in $L^2(\Lambda_{2L})$, this implies that, for sufficiently small E,

$$\|\psi\|_L \geq \frac{1}{2}\|\varphi\|_L \quad \text{and} \quad \|\varphi - \overline{e_\theta}\psi\|_L \leq e^{-L/C}\|\psi\|_L. \tag{4.18}$$

As V_ω^{per} is uniformly bounded, it follows from our choice for L and l and estimate (4.18) that, for E sufficiently small,

$$\begin{aligned} \int_{\Lambda_{2l}} |\psi(x)|^2 dx &\geq \frac{1}{C_1}\left(\frac{l}{L}\right)^2 \int_{\Lambda_{2L}} |\varphi(x)|^2 dx - C\|\varphi - \overline{e_\theta}\psi\|_L^2 \\ &\geq \frac{1}{\tilde{C}_1}\left(\frac{l}{L}\right)^2 \int_{\Lambda_{2L}} |\psi(x)|^2 dx, \end{aligned}$$

$$\int_{\Lambda_{2l}} V_\omega^{\mathrm{per}}(x)|\psi(x)|^2 dx = \int_{\Lambda_{2l}} V_\omega^{\mathrm{per}}(x)|\varphi(x)|^2 dx + C\|\varphi - \overline{e_\theta}\psi\|_L^2 \leq \tilde{C}_2 E \int_{\Lambda_{2l}} |\psi(x)|^2 dx.$$

Hence, we obtain inequalities (4.15) - (4.16) with φ replaced by $\psi \in P_q L^2(\mathbb{R}^2)$. Now, we write $\psi = \sum_{j\geq 0} a_j e_j$ (see (3.26)). Using the fact that $\{e_j\}_{j\geq 0}$ is an orthogonal family on any disk centered at 0 (this is due to the rotational symmetry), we compute

$$\int_{\Lambda_{2l}} |\psi(x)|^2 dx \leq \int_{|x|\leq\sqrt{2}l} |\psi(x)|^2 dx = \sum_{j\geq 0} |a_j|^2 \int_{|x|\leq\sqrt{2}l} |e_j(x)|^2 dx, \tag{4.19}$$

and

$$\int_{\Lambda_{2L}} |\psi(x)|^2 dx \geq \int_{|x|\leq L} |\psi(x)|^2 dx = \sum_{j\geq 0} |a_j|^2 \int_{|x|\leq L} |e_j(x)|^2 dx. \tag{4.20}$$

Fix $m \geq 1$ and decompose $\psi = \psi_0 + \psi_m$ where

$$\psi_0 = \sum_{j=0}^{m} a_j e_j, \qquad \psi_m = \sum_{j\geq m+1} a_j e_j. \tag{4.21}$$

Our next goal is to estimate the ratio

$$\frac{\int_{|x|<\sqrt{2}l} |e_{j,q}(x)|^2 dx}{\int_{|x|<L} |e_{j,q}(x)|^2 dx}, \quad j \geq m+1, \tag{4.22}$$

where l, m, and L satisfy (4.6) with suitable $\mathcal{C}$, under the hypotheses that l, and hence m and L are sufficiently large. Passing to polar coordinates (r, θ), and then changing the variable $s = \frac{b\rho^2}{2j}$ in both the numerator and the denominator of (4.22), we find that

$$\frac{\int_{|x|<\sqrt{2}l} |e_{j,q}(x)|^2 dx}{\int_{|x|<L} |e_{j,q}(x)|^2 dx} = \frac{\int_0^{bl^2/j} e^{-s(j-q)} s^{j-q} L_q^{(j-q)}(js)^2 ds}{\int_0^{bL^2/(2j)} e^{-s(j-q)} s^{j-q} L_q^{(j-q)}(js)^2 ds}. \tag{4.23}$$

Employing estimates (3.27) and (3.28), we get

$$\frac{\int_0^{bl^2/j} e^{-s(j-q)} s^{j-q} L_q^{(j-q)}(js)^2 ds}{\int_0^{bL^2/(2j)} e^{-s(j-q)} s^{j-q} L_q^{(j-q)}(js)^2 ds} \leq C(q) \left(\frac{j}{j-q}\right)^{2q} \frac{\int_0^{bl^2/j} e^{(j-q)f(s)} ds}{\int_0^{\epsilon(j)} e^{(j-q)f(s)} ds} \tag{4.24}$$

where

$$f(s) := \ln s - s, \quad s > 0,$$

and

$$\epsilon(j) = \begin{cases} \frac{1}{2} & \text{if} \quad j \leq bL^2, \\ \frac{bL^2}{2j} & \text{if} \quad j > bL^2. \end{cases}$$

Note that the function f is increasing on the interval $(0, 1)$. Since $j \geq m + 1$, and $\mathcal{C}$, the constant in (4.6), is greater than one, we have $\frac{bl^2}{j} < 1$. Therefore,

$$\int_0^{bl^2/j} e^{(j-q)f(s)} ds \leq \frac{bl^2}{j} e^{(j-q)f(bl^2/j)}. \tag{4.25}$$

On the other hand, using a second-order Taylor expansion of f, we get

$$f(s) \geq f(\epsilon(j)) + \frac{s - \epsilon(j)}{\epsilon(j)} - \frac{1}{2}, \quad s \in (\epsilon(j), \epsilon(j)/2).$$

Consequently,

$$\int_0^{\epsilon(j)} e^{(j-q)f(s)} ds \geq \int_{\epsilon(j)/2}^{\epsilon(j)} e^{(j-q)f(s)} ds \geq \frac{\epsilon(j)}{2} e^{(j-q)(f(\epsilon(j))-1))}. \tag{4.26}$$

Putting together (4.24) - (4.26), we obtain

$$\begin{aligned} \frac{\int_{|x|<\sqrt{2}l} |e_{j,q}(x)|^2 dx}{\int_{|x|<n} |e_{j,q}(x)|^2 dx} &\leq C(q) \frac{2bl^2}{j\epsilon(j)} \left(\frac{j}{j-q}\right)^{2q} \exp\left((j-q)(f(bl^2/j) - f(\epsilon(j)) + 1\right) \\ &\leq \tilde{C}(q) \frac{2bl^2}{j\epsilon(j)} \left(\frac{j}{j-q}\right)^{2q} \begin{cases} j^q \exp\left(-bl^2 + j \ln\left(\frac{2e^{3/2}bl^2}{j}\right)\right) & \text{if} \quad j < bL^2, \\ \exp\left(-bl^2 + j \ln\left(\frac{2e^2 l^2}{L^2}\right)\right) & \text{if} \quad j \geq bL^2. \end{cases} \end{aligned} \tag{4.27}$$

Now, using the computations (4.19) and (4.20) done for ψ_m, as well as (4.6), we obtain

$$\begin{aligned} \int_{\Lambda_{2l}} |\psi_m(x)|^2 dx &\leq C e^{-bl^2/2 + m \ln(\mathcal{C}bl^2/2m)} \int_{\Lambda_{2L}} |\psi(x)|^2 dx \\ &\leq C_1 \left(\frac{L}{l}\right)^2 e^{-bl^2/2 + m \ln(\mathcal{C}bl^2/m)} \int_{\Lambda_{2l}} |\psi(x)|^2 dx. \end{aligned} \tag{4.28}$$

Plugging this into (4.15) – (4.16), and using the uniform boundedness of V_ω, we get that

$$\int_{\Lambda_{2l}} V_\omega(x) |\psi_0(x)|^2 dx \leq \left(C_2 E + C \left(\frac{L}{l}\right)^2 e^{-bl^2 + m \ln(\mathcal{C}bl^2/2m)}\right) \int_{\Lambda_{2l}} |\psi_0(x)|^2 dx,$$

$$2 \int_{\Lambda_{2l}} |\psi_0(x)|^2 dx \geq \left(\frac{1}{C_1} \left(\frac{l}{L}\right)^2 - e^{-bl^2 + m \ln(\mathcal{C}bl^2/2m)}\right) \int_{\Lambda_{2L}} |\psi(x)|^2 dx.$$

Taking (4.7) into consideration, this completes the proof of Lemma 4.1. $\square$

Let us now complete the proof of Theorem 4.1. Assume at first the hypotheses of its first part. In particular, suppose that $u(x) \geq C(1 + |x|)^{-\varkappa}$, $x \in \mathbb{R}^2$, with some $\varkappa > 2$, and $C > 0$. Pick $\eta > 2/(\varkappa - 2)$, and $\nu_0 > \max\left\{\frac{1}{\varkappa-2}, \nu\right\}$ where $\nu = \nu(\eta)$ is the number defined in Corollary 3.1. Finally, fix an arbitrary $\varkappa' > \varkappa$ and set

$$n \sim E^{-\nu_0}, \quad L = (2n+1)a/2, \quad l = E^{-\frac{1}{\varkappa'-2}}, \quad m \sim E^{-\frac{2}{\varkappa-2}}.$$

Then the numbers m, l, and L, satisfy (4.6) – (4.7) provided that $E > 0$ is sufficiently small. Further, for any $\gamma_0 \in l\mathbb{Z}^2 \cap \Lambda_{2L}$ we have

$$\langle V_\omega^{\gamma_0} \psi, \psi \rangle_l \geq \sum_{|\gamma| \leq l} \omega_\gamma \int_{\Lambda_{2l}} u(x - \gamma) |\psi(x)|^2 dx \geq \frac{1}{C_3} l^{-\varkappa} \sum_{|\gamma| \leq l} \omega_\gamma \int_{\Lambda_{2l}} |\psi(x)|^2 dx \tag{4.29}$$

with $C_3 > 0$ independent of θ and E. Hence, the probability that there exists $\gamma \in 2l\mathbb{Z}^2 \cap \Lambda_{2L}$ and a non identically vanishing function ψ in the span of $\{e_j\}_{0\leq j\leq m}$ such that (4.8) be satisfied, is not greater than the probability that

$$l^{-2} \sum_{|\gamma|\leq l} \omega_\gamma \leq C_3 E l^{\varkappa-2} = C_3 E^{\frac{\varkappa'-\varkappa}{\varkappa'-2}}. \tag{4.30}$$

Applying a standard large-deviation estimate (see e.g. [15, Subsection 8.4] or [22, Section 3.2]), we easily find that the probability that (4.30) holds, is bounded by

$$\exp\left(C_4 l^2 \ln \mathbb{P}(\omega_0 \leq C_3 E^{\frac{\varkappa'-\varkappa}{\varkappa'-2}})\right) = \exp\left(C_4 E^{\frac{2}{\varkappa'-2}} \ln \mathbb{P}(\omega_0 \leq C_3 E^{\frac{\varkappa'-\varkappa}{\varkappa'-2}})\right)$$

with C_4 independent of θ and $E > 0$ small enough. Applying our hypothesis that $\mathbb{P}(\omega_0 \leq E) \sim CE^\kappa$, $E \downarrow 0$, with $C > 0$ and $\kappa > 0$, we find that for any $\varkappa' > \varkappa$, $\theta \in \mathbb{T}^*_{2L}$, and sufficiently small $E > 0$, we have

$$\mathbb{P}(r_{q,n,\omega}(\theta) \text{ has an eigenvalue less than } E) \leq \exp\left(-C_5 E^{\frac{2}{\varkappa'-2}} |\ln E|\right) \tag{4.31}$$

with $C_5 > 0$ independent of θ and E. Putting together (3.39), (4.5) and (4.31), and taking into account that area $\Lambda^*_{2L} = \pi^2 L^{-2}$, we get

$$\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq) \leq \frac{b}{2\pi} \exp\left(-C_5 E^{\frac{2}{\varkappa'-2}} |\ln E|\right) + \exp\left(-E^{-\eta}\right)$$

which implies

$$\liminf_{E\downarrow 0} \frac{\ln |\ln \mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq)|}{|\ln E|} \geq \frac{2}{\varkappa'-2}$$

for any $\varkappa' > \varkappa$. Letting $\varkappa' \downarrow \varkappa$, we get (4.1).

Assume now the hypotheses of Theorem 4.1 ii). In particular, we suppose that $u(x) \geq Ce^{-C|x|^\beta}$, $x \in \mathbb{R}^2$, $C > 0$, $\beta > 0$. Put $\beta_0 = \max\{1, 2/\beta\}$. Pick an arbitrary $\beta' > \beta$ and set

$$l = |\ln E|^{1/\beta'}, \quad m \sim |\ln E|^{\beta_0}.$$

Then (4.6) - (4.7) are satisfied provided that $E > 0$ is sufficiently small, and similarly to (4.29), for any $\gamma_0 \in 2l\mathbb{Z}^2 \cap \Lambda_{2L}$ we have

$$\langle V_\omega^{\gamma_0} \psi, \psi\rangle_l \geq \frac{1}{C_6} e^{-c_6 l^\beta} \sum_{|\gamma|\leq l} \omega_\gamma \int_{\Lambda_{2l}} |\psi(x)|^2 dx$$

with $C_6 > 0$ independent of θ and E. Arguing as in the derivation of (4.31), we get

$$\mathbb{P}(r_{q,n,\omega}(\theta) \text{ has an eigenvalue less than } E) \leq \exp\left(-C_7 |\ln E|^{1+2/\beta'} \ln |\ln E|\right) \tag{4.32}$$

with $C_7 > 0$ independent of θ and E. As in the previous case, we put together (3.39), (4.5) and (4.31), and obtain the estimate

$$\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq) \leq \frac{b}{2\pi} \exp\left(-C_7 |\ln E|^{1+2/\beta'} \ln|\ln E|\right) + \exp\left(-E^{-\eta}\right)$$

which implies

$$\liminf_{E\downarrow 0} \frac{\ln|\ln \mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq)|}{\ln|\ln E|} \geq 1 + \frac{2}{\beta'}$$

for any $\beta' > \beta$. Letting $\beta' \downarrow \beta$, we get (4.2).
Finally, let us assume the hypotheses of Theorem 4.1 iii). In particular, we assume that $u(x) \geq C\mathbf{1}_{\{x\in\mathbb{R}^2; |x-x_0|<\varepsilon\}}$ with some $C > 0$, $x_0 \in \mathbb{R}^2$, and $\varepsilon > 0$. Due to $\tau_{x_0} H_0 \tau_{x_0}^* = H_0$ and $\tau_{x_0}\mathbf{1}_{\{x\in\mathbb{R}^2;|x-x_0|<\varepsilon\}}\tau_{x_0}^* = \mathbf{1}_{\{x\in\mathbb{R}^2;|x|<\varepsilon\}}$ we can assume without loss of generality that $x_0 = 0$. Our first goal is to estimate from below the ratio

$$R_\gamma = R_{\gamma,m,q} := \frac{\int_{|x-\gamma|\leq\varepsilon} |\mathcal{P}_m(x)|^2 dx}{\int_{|x|\leq\sqrt{2}l} |\mathcal{P}_m(x)|^2 dx} \tag{4.33}$$

where

$$\mathcal{P}_m(x) := \sum_{j=0}^{q} c_j e_{j,q}(x), \quad x \in \mathbb{R}^2, \tag{4.34}$$

with $0 \neq \mathbf{c} = (c_0, c_1, \ldots, c_m) \in \mathbb{C}^m$.

Lemma 4.2. *Let $q \in \mathbb{Z}_+$. Let $\pi(s) = \sum_{j=0}^q c_j s^j$, $s \in \mathbb{R}$. Moreover, let $p \in \mathbb{Z}_+$, $\rho \in (0,\infty)$. Then we have*

$$\left(\prod_{r=0}^{q}(r!)^2\right) \frac{e^{-(q+1)\rho}\rho^{q(q+1)}}{(1+\rho^q)^q} \frac{\rho^{p+1}}{(p+2q+1)^{(q+1)^2-q}}|\mathbf{c}|^2$$
$$\leq \int_0^\rho |\pi(s)|^2 e^{-s} s^p ds \leq (1+\rho^q)\frac{\rho^{p+1}}{p+1}|\mathbf{c}|^2 \tag{4.35}$$

where $\mathbf{c} := (c_0, c_1, \ldots, c_q) \in \mathbb{C}^{q+1}$ and $|\mathbf{c}|^2 = |c_0|^2 + \cdots + |c_q|^2$.

Proof. Let $\mathcal{M}$ be the $(q+1)\times(q+1)$ positive-definite matrix with entries $\int_0^\rho s^{j+k+p}e^{-s}ds$, $j,k = 0,1,\ldots,q$. Then we have

$$\int_0^\rho |\pi(s)|^2 e^{-s}s^p ds = \langle \mathcal{M}\mathbf{c}, \mathbf{c}\rangle \leq \|\mathcal{M}\||\mathbf{c}|^2.$$

Further, $\mathcal{M} = \int_0^\rho \mathcal{E}(s)ds$ where $\mathcal{E}(s)$, $s \in (0,\rho)$, is the rank-one matrix with entries $s^{j+k+p}e^{-s}s^p$, $j,k = 0,1,\ldots,q$. Obviously,

$$\|\mathcal{E}(s)\| = \sqrt{\sum_{j=0}^{q} s^{2j}}\, e^{-s}s^p \leq (1+s^q)e^{-s}s^p, \quad s \in (0,\rho),$$

and

$$\|\mathcal{M}\| \leq \int_0^\rho \|\mathcal{E}(s)\| ds \leq \int_0^\rho (1+s^q)e^{-s}s^p ds \leq \frac{\rho^{p+1}(1+\rho^q)}{p+1}$$

which yields the upper bound in (4.35). Next, we have

$$\frac{\det \mathcal{M}}{\|\mathcal{M}\|^q}|\mathbf{c}|^2 \leq \int_0^\rho |\pi(s)|^2 e^{-s}s^p ds. \tag{4.36}$$

Further,

$$e^{-(1+q)\rho} \det \tilde{\mathcal{M}} \leq \det \mathcal{M} \tag{4.37}$$

where $\tilde{\mathcal{M}}$ is the $(q+1)\times(q+1)$-matrix with entries $\int_0^\rho s^{j+k+p}ds = \frac{\rho^{j+k+p+1}}{j+k+p+1}$, $j,k = 0,1,\ldots,q$, and

$$\det \tilde{\mathcal{M}} = \rho^{q(q+1)}\Delta_q \tag{4.38}$$

where $\Delta_q = \Delta_q(p)$ is the determinant of the $(q+1)\times(q+1)$-matrix with entries $(j+k+p+1)^{-1}$, $j,k = 0,1,\ldots,q$. On the other hand, it is easy to check that

$$\Delta_q = \frac{(q!)^2}{(p+2q+1)\prod_{r=0}^{q-1}(p+q+r+1)^2}\Delta_{q-1}, \quad q \geq 1,\ p \geq 0, \quad \Delta_0 = \frac{1}{p+1}.$$

Hence, for $q \geq 1$ and $p \geq 0$

$$\frac{\prod_{r=0}^{q}(r!)^2}{(p+2q+1)^{(q+1)^2}} \leq \Delta_q. \tag{4.39}$$

Putting together (4.36) – (4.39) and using the upper bound in (4.35), we obtain the corresponding lower bound. ☐

In the following proposition we obtain the needed lower bound of ratio (4.33).

Proposition 4.1. *There exists a constant $\mathcal{C} > 0$ such that for sufficiently large m and l ratio (4.33) satisfies the estimates*

$$R_\gamma \geq e^{-\mathcal{C}m\ln l} \tag{4.40}$$

for each linear combination $\mathcal{P}_m$ of the form (4.34).

Proof. Evidently,

$$\int_{|x-\gamma|\leq\varepsilon} |\mathcal{P}_m(x)|^2 dx = \int_{|x|\leq\varepsilon} |\mathcal{P}_m(x+\gamma)|^2 dx = \int_{|x|\leq\varepsilon} |(\tau_\gamma\mathcal{P}_m)(x)|^2, \tag{4.41}$$

$$\int_{|x|\leq\sqrt{2}l} |\mathcal{P}_m(x)|^2 dx \leq \int_{|x-\gamma|\leq 2\sqrt{2}l} |\mathcal{P}_m(x)|^2 =$$

$$\int_{|x|\leq 2\sqrt{2}l} |\mathcal{P}_m(x+\gamma)|^2 dx = \int_{|x|\leq 2\sqrt{2}l} |(\tau_\gamma\mathcal{P}_m)(x)|^2 dx, \tag{4.42}$$

the magnetic translation operator τ_γ being defined in (2.6). Using the fact that τ_γ commutes with the the creation operator a^* (see (3.30)), we easily find that (3.29) implies

$$(\tau_\gamma \mathcal{P}_m)(x) = \sum_{j=0}^{m} \tilde{c}_j (a^*)^q \left(z^j e^{\zeta z} e^{-b|z|^2/4} \right) \tag{4.43}$$

where $z = x_1 + ix_2$, $\zeta = -\frac{b}{2}(\gamma_1 - i\gamma_2)$, and the coefficients $\tilde{c}_j$, $j = 0, 1, \ldots, m$, may depend on γ, b and q but are independent of $x \in \mathbb{R}^2$. Applying (3.26) and (3.29), we get

$$\sum_{j=0}^{m} \tilde{c}_j (a^*)^q \left(z^j e^{\zeta z} e^{-b|z|^2/4} \right) = \sum_{j=0}^{m} \tilde{c}_j \sum_{k=0}^{\infty} \frac{\zeta^k}{k!} (a^*)^q \left(z^{j+k} e^{-b|z|^2/4} \right) =$$

$$e^{-b|z|^2/4} \sum_{j=0}^{m} \hat{c}_j z^{j-q} \sum_{k=0}^{\infty} \frac{(\zeta z)^k}{k!} L_q^{(j+k-q)}(b|z|^2/2) \tag{4.44}$$

with $\hat{c}_j$, $j = 0, 1, \ldots, m$, independent of $x \in \mathbb{R}^2$. By [9, Eq.(8.977.2)] we have

$$\sum_{k=0}^{\infty} \frac{(\zeta z)^k}{k!} L_q^{(j+k-q)}(b|z|^2/2) = e^{\zeta z} L_q^{(j-q)} \left(\frac{b|z|^2}{2} - \zeta z \right), \tag{4.45}$$

while the Taylor expansion formula entails

$$L_q^{(j-q)} \left(\frac{b|z|^2}{2} - \zeta z \right) = \sum_{s=0}^{q} \frac{(-\zeta z)^s}{s!} \frac{d^s L_q^{(j-q)}(\xi)}{d\xi^s}_{\big|\xi=b|z|^2/2}, \tag{4.46}$$

and [9, Eq.(8.971.3)] yields

$$\frac{d^s L_q^{(j-q)}(\xi)}{d\xi^s} = (-1)^s L_{q-s}^{(j-q+s)}(\xi), \quad \xi \in \mathbb{R}. \tag{4.47}$$

Combining (4.43) - (4.47), we find that

$$(\tau_\gamma \mathcal{P}_m)(x) = e^{\zeta z} \tilde{\mathcal{P}}_m(x), \quad x \in \mathbb{R}^2, \tag{4.48}$$

where

$$\tilde{\mathcal{P}}_m(x) = e^{-b|z|^2/4} \sum_{j=0}^{m} \hat{c}_j \sum_{s=0}^{q} \frac{\zeta^s}{s!} z^{j+s-q} L_{q-s}^{(j+s-q)}(b|z|^2/2) =$$

$$e^{-b|z|^2/4} \sum_{p=0}^{m+q} z^{p-q} \phi_{p,q}(b|z|^2/2), \tag{4.49}$$

and $\phi_{p,q}$, $p = 0, \ldots, m+q$, are polynomials of degree not exceeding q; moreover, if $p < q$, then the minimal possible degree of the non-zero monomial terms in $\phi_{p,q}$, is $q - p$.

Bearing in mind that $|e^{\zeta z}|^2 = e^{x\cdot\gamma}$ and $|\gamma| \leq \frac{\sqrt{2}}{2} l$, we find that there exists a constant C such that for sufficiently large l we have

$$R_\gamma \geq e^{-Cl^2} \tilde{R} \tag{4.50}$$

where

$$\tilde{R} = \frac{\int_{|x|\leq\varepsilon} |\tilde{\mathcal{P}}_m(x)|^2 dx}{\int_{|x|\leq 2\sqrt{2}l} |\tilde{\mathcal{P}}_m(x)|^2 dx}, \tag{4.51}$$

the functions $\tilde{\mathcal{P}}_m$ being defined in (4.49). Passing to the polar coordinates (r, θ) in $\mathbb{R}^2$, after that changing the variable $s = br^2/2$, and taking into account the rotational symmetry we find that for each $R > 0$ we have

$$\int_{|x|\leq R} |\mathcal{P}_m(x)|^2 dx = \frac{2\pi}{b} \sum_{p=0}^{m+q} \left(\frac{2}{b}\right)^{p-q} \int_0^\rho s^{p-q} e^{-s} |\phi_{p,q}(s)|^2 ds =$$

$$\sum_{p=0}^{m} \int_0^\rho s^p e^{-s} |\Pi_{p,q}(s)|^2 ds + \sum_{p=1}^{q} \int_0^\rho s^p e^{-s} |\tilde{\Pi}_{p,q}(s)|^2 ds; \tag{4.52}$$

if $q = 0$, then the second term in the last line of (4.52) should be set equal to zero. Here $\rho = bR^2/2$, $\Pi_{p,q}(s) = \sqrt{\frac{2\pi}{b}\left(\frac{2}{b}\right)^p} \phi_{p+q,q}(s)$, $p = 0, \ldots, m$, $\tilde{\Pi}_{p,q} = \sqrt{\frac{2\pi}{b}\left(\frac{2}{b}\right)^{-p}} s^{-p} \phi_{q-p,q}(s)$, $p = 1, \ldots, q$. Note that the degree of the polynomials $\Pi_{q,p}$ does not exceed q, and the the degree of the polynomials $\tilde{\Pi}_{q,p}$ does not exceed $q - p$. Bearing in mind (4.52) and applying Lemma 4.2, we easily deduce the existence of a constant $C > 0$ such that for sufficiently large m and l we have

$$\tilde{R} \geq e^{-Cm \ln l},$$

which combined with (4.50) yields (4.40). □

Next, we pick an arbitrary η and $\nu = \nu(\eta)$, the number defined in Corollary 3.1. Further, we choose $\varsigma > 1$ and $\delta \in (0, 1/2)$ so that $\varsigma(1 - \delta) > 1 + 2\nu$, and set

$$l = |\ln E|^{\delta/2}, \quad m \sim \frac{\varsigma |\log E|}{\log |\log E|}, \quad L = (2n+1)a/2. \tag{4.53}$$

Then, for E sufficiently small, (4.6) – (4.7) are satisfied. Further, we impose the additional condition that $\mu := \frac{\mathcal{C}\varsigma\delta}{2} < 1$ where $\mathcal{C}$ is the constant in (4.40), which is compatible with the conditions on ς and δ formulated above. Now, the probability that there exists $\gamma \in 2l\mathbb{Z}^2 \cap \Lambda_{2L}$ and a non identically vanishing function ψ in the span of $\{e_j\}_{0\leq j\leq m}$ such that (4.8) be satisfied, is not greater than the probability that

$$l^{-2} \sum_{|\gamma|\leq l} \omega_\gamma \leq l^{-2} E^{1-\mu} = E^{1-\mu} |\ln E|^{\delta}.$$

Arguing as in the derivation of (4.31) and (4.32), we conclude that for any $\theta \in \mathbb{T}_{2L}^*$ we have

$$\mathbb{P}(r_{q,m,\omega} \text{ has an eigenvalue less than } E) \leq$$

$$\exp\left(C_8 l^2 \log \mathbb{P}(\omega_0 \leq E^{1-\mu}|\ln E|^\delta\right) \leq \exp\left(-C_9|\ln E|^{1+\delta}\ln|\ln E|\right) \tag{4.54}$$

with positive C_8 and C_9 independent of θ and $E > 0$ small enough. Combining the upper bound in (3.39), (4.5), and (4.54), we get (4.3).
This completes the proof of the upper bounds in Theorem 2.1.

5 Proof of Theorem 2.1: lower bounds of the IDS

In this section we get the lower bounds of $\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq)$ needed for the proof of Theorem 2.1.

Theorem 5.1. *Assume that* $\mathbf{H}_1$ *–* $\mathbf{H}_4$ *hold, that almost surely* $\omega_\gamma \geq 0$, $\gamma \in \mathbb{Z}^2$, *and* (2.1) *is valid. Fix the Landau level* $2bq$, $q \in \mathbb{Z}_+$.
i) We have

$$\liminf_{E\downarrow 0} \frac{\ln|\ln(\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq))|}{|\ln E|} \leq \frac{2}{\varkappa - 2}, \tag{5.1}$$

where $\varkappa$ *is the constant in* (1.2).
ii) Let $u(x) \leq e^{-C|x|^\beta}$, $x \in \mathbb{R}^2$, *for some* $C > 0$ *and* $\beta \in (0, 2]$. *Then we have*

$$\limsup_{E\downarrow 0} \frac{\ln(\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq))}{|\ln E|^{1+2/\beta}} \geq -\frac{\pi\kappa}{C}, \tag{5.2}$$

if $\beta \in (0, 2)$, *and*

$$\liminf_{E\downarrow 0} \frac{\ln(\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq))}{|\ln E|^2} \geq -\pi\kappa\left(\frac{2}{b} + \frac{1}{C}\right), \tag{5.3}$$

if $\beta = 2$. *Therefore,*

$$\limsup_{E\downarrow 0} \frac{\ln|\ln(\mathcal{N}_b(2bq+E) - \mathcal{N}_b(2bq))|}{\ln|\ln E|} \leq 1 + 2/\beta. \tag{5.4}$$

Note that the combination of Theorem 4.1 with Theorem 5.1 completes the proof of Theorem 2.1.
Let us prove now Theorem 5.1. Pick $\eta \geq \frac{2}{\varkappa-2}$ in the case of its first part, or an arbitrary $\eta > 0$ in the case of its second part. As above, set $n \sim E^{-\nu}$ where $\nu = \nu(\eta)$ is the number defined in Corollary 3.1, and $L = (2n+1)a/2$. Bearing in mind the lower

bound in (3.39), and (4.4), we conclude that it suffices to estimate from below the quantity

$$\begin{aligned}\mathbb{E}\left(\rho_{q,n,\omega}(E)\right) &= \frac{1}{(2\pi)^2}\int_{\Lambda_{2L}^*}\mathbb{E}(N(E;r_{q,n,\omega}(\theta))d\theta \\ &= (2\pi)^{-2}\int_{\Lambda_{2L}^*}\sum_{j=1}^{\operatorname{rank} r_{q,n,\omega}(\theta)}\mathbb{P}(\lambda_j(\theta)<E)d\theta \\ &\geq (2\pi)^{-2}\int_{\Lambda_{2L}^*}\mathbb{P}(\lambda_1(\theta)<E)d\theta.\end{aligned} \tag{5.5}$$

Fix an arbitrary $\theta \in \mathbb{T}_{2L}^*$. Evidently, $\mathbb{P}(\lambda_1(\theta) < E)$ is equal to the probability that there exists a non-zero function $f \in \operatorname{Ran} r_{q,n,\omega}(\theta)$ such that

$$\int_{\Lambda_{2L}} V_\omega(x)|f(x;\theta)|^2dx < E\int_{\Lambda_{2L}}|f(x;\theta)|^2dx. \tag{5.6}$$

Further, pick the trial function

$$\varphi(x;\theta) = \sum_{\gamma\in 2L\mathbb{Z}^2} e^{-i\theta(x+\gamma)}(\tau_\gamma\tilde{\varphi})(x), \quad x\in\Lambda_{2L}, \quad \theta\in\mathbb{T}_{2L}^*, \tag{5.7}$$

where

$$\tilde{\varphi}(x) = \tilde{\varphi}_q(x) := \overline{z}^q e^{-b|z|^2/4}, \quad z = x_1+ix_2, \quad \overline{z} = x_1 - ix_2. \tag{5.8}$$

Since the function $\tilde{\varphi}_q$ is proportional to $e_{0,q}$ (see (3.26)), we have $\varphi \in \operatorname{Ran} r_{q,n,\omega}(\theta)$. Therefore, the probability that there exists a non-zero function $f \in \operatorname{Ran} r_{q,n,\omega}(\theta)$ such that (5.6) holds, is not less than the probability that

$$\int_{\Lambda_{2L}} V_\omega(x)|\varphi(x;\theta)|^2dx < E\int_{\Lambda_{2L}}|\varphi(x;\theta)|^2dx. \tag{5.9}$$

Lemma 5.1. *Let the function φ be defined as in (5.7) – (5.8). Then there exist $L_0 > 0$ and $c_1 > 0$ independent of θ such that $L \geq L_0$ implies*

$$\int_{\Lambda_{2L}}|\varphi(x;\theta)|^2dx > c_1. \tag{5.10}$$

Proof. We have $\varphi = \varphi_0 + \varphi_\infty$ where

$$\varphi_0(x;\theta) = e^{-i\theta x}\tilde{\varphi}(x), \tag{5.11}$$

$$\varphi_\infty(x;\theta) = \sum_{\gamma\in 2L\mathbb{Z}^2,\, \gamma\neq 0} e^{-i\theta(x+\gamma)}(\tau_\gamma\tilde{\varphi})(x). \tag{5.12}$$

Note that

$$\sup_{x\in\Lambda_{2L}} |\varphi_\infty(x;\theta)| \leq \tilde{c}e^{-\tilde{c}L^2} \tag{5.13}$$

with $\tilde{c}$ independent of L and θ. Further,

$$\begin{aligned}\int_{\Lambda_{2L}} |\varphi(x;\theta)|^2 dx &\geq \frac{1}{2}\int_{\Lambda_{2L}} |\varphi_0(x;\theta)|^2 dx - 2\int_{\Lambda_{2L}} |\varphi_\infty(x;\theta)|^2 dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} |\tilde{\varphi}(x)|^2 dx - 8\tilde{c}L^2 e^{-\tilde{c}L^2}.\end{aligned} \tag{5.14}$$

Taking into account that $\int_{\mathbb{R}^2} |\tilde{\varphi}|^2 dx = \frac{2\pi}{b}\left(\frac{2}{q}\right)^q q!$, we find that (5.14) implies (5.10). □

By assumption we have

$$u(x) \leq Cv(x), \quad C > 0, \quad x \in \mathbb{R}^2, \tag{5.15}$$

where $v(x) := (1+|x|)^{-\varkappa}$ in the case of Theorem 5.1 i), and $v(x) := e^{-C|x|^\beta}$ in the case of Theorem 5.1 ii). Since $\omega_\gamma \geq 0$, inequality (5.9) will follow from

$$\sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma \int_{\Lambda_{2L}} v(x-\gamma)|\varphi(x;\theta)|^2 dx \leq c_2 E \tag{5.16}$$

where $c_2 = c_1 C^{-1}$, C being the constant in (5.15), and c_1 being the constant in (5.10). Next, we write

$$\sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma \int_{\Lambda_{2L}} v(x-\gamma)|\varphi(x;\theta)|^2 dx \leq$$

$$2\sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma \int_{\Lambda_{2L}} v(x-\gamma)|\varphi_0(x;\theta)|^2 dx + 2\sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma \int_{\Lambda_{2L}} v(x-\gamma)|\varphi_\infty(x;\theta)|^2 dx \tag{5.17}$$

where φ_0 and φ_∞ are defined in (5.11) and (5.12) respectively.

Lemma 5.2. *Fix $q \in \mathbb{Z}_+$.*
i) Let $\varkappa > 0$, $b > 0$. Then there exists a constant $c' > 0$ such that for each $y \in \mathbb{R}^2$, $L > 0$, and $\theta \in \mathbb{T}^*_{2L}$, we have

$$\int_{\Lambda_{2L}} (1+|x-y|)^{-\varkappa}|\varphi_0(x;\theta)|^2 dx \leq c'(1+|y|)^{-\varkappa}. \tag{5.18}$$

ii) Let $\beta \in (0,2]$, $b > 0$, $C > 0$. If $\beta \in (0,2)$, set $b_0 := C$. If $\beta = 2$, set $b_0 := \frac{Cb}{2C+b}$. Then for each $b_1 < b_0$ there exists a constant $c'' > 0$ such that for each $y \in \mathbb{R}^2$, $L > 0$, and $\theta \in \mathbb{T}^*_{2L}$, we have

$$\int_{\Lambda_{2L}} e^{-C|x-y|^\beta}|\varphi_0(x;\theta)|^2 dx \leq c'' e^{-b_1|y|^\beta}. \tag{5.19}$$

We omit the proof since estimates (5.18) – (5.19) follow from standard simple facts concerning the asymptotics at infinity of the convolutions of functions admitting power-like or exponential decay, with the derivatives of Gaussian functions. In the case of power-like decay, results of this type can be found in in [34, Theorem 24.1], and in the case of an exponential decay similar results are contained in [12, Lemma 3.5].
Using Lemma 5.2, we find that under the hypotheses of Theorem 5.1 i) we have

$$2\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma\int_{\Lambda_{2L}} v(x-\gamma)|\varphi_0(x;\theta)|^2dx \le c_3\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma(1+\gamma|)^{-\varkappa}, \tag{5.20}$$

while under the hypotheses of Theorem 5.1 ii) for each $b_1 < b_0$ we have

$$2\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma\int_{\Lambda_{2L}} v(x-\gamma)|\varphi_0(x;\theta)|^2dx \le c_3\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma e^{-b_1|\gamma|^2}, \tag{5.21}$$

where c_3 is independent of L and θ. Further, for both parts of Theorem 5.1 we have

$$2\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma\int_{\Lambda_{2L}} v(x-\gamma)|\varphi_\infty(x;\theta)|^2dx \le c_4L^2e^{-\tilde{c}L^2} \tag{5.22}$$

where c_4 is independent of L and θ, and $\tilde{c}$ is the constant in (5.13). Since $L \sim E^{-\nu}$, $\nu > 0$, we have

$$c_2E - c_4L^2e^{-\tilde{c}L^2} \ge \frac{c_2}{2}E \tag{5.23}$$

for sufficiently small $E > 0$. Combining (5.17) with (5.20) – (5.23), and setting $c_5 = c_2/(2c_3)$, we find that (5.16) will follow from the inequality

$$\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma c_5(1+|\gamma|)^{-\varkappa} \le c_5E, \tag{5.24}$$

in the case of Theorem 5.1 i), or from the inequality

$$\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma e^{-b_1|\gamma|^\beta} \le c_5E, \quad b_1 < b_0, \tag{5.25}$$

in the case of Theorem 5.1 ii). Now pick $l > 0$ and write

$$\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma(1+|\gamma|)^{-\varkappa} \le \sum_{\gamma\in\mathbb{Z}^2,\,|\gamma|\le l}\omega_\gamma + \sum_{\gamma\in\mathbb{Z}^2,\,|\gamma|>l}\omega_\gamma|\gamma|^{-\varkappa}, \tag{5.26}$$

$$\sum_{\gamma\in\mathbb{Z}^2}\omega_\gamma e^{-b_1|\gamma|^\beta} \le \sum_{\gamma\in\mathbb{Z}^2,\,|\gamma|\le l}\omega_\gamma + \sum_{\gamma\in\mathbb{Z}^2,\,|\gamma|>l}\omega_\gamma e^{-b_1|\gamma|^\beta}. \tag{5.27}$$

Evidently, for each $\varkappa' \in (2,\varkappa)$ and $b_2 < b_1$ there exists a constant $c_6 > 0$ such that

$$\sum_{\gamma\in\mathbb{Z}^2,\,|\gamma|>l}\omega_\gamma|\gamma|^{-\varkappa} \le c_6l^{-\varkappa'+2}, \tag{5.28}$$

$$\sum_{\gamma\in\mathbb{Z}^2,\,|\gamma|>l} \omega_\gamma e^{-b_1|\gamma|^\beta} \leq c_6 e^{-b_2 l^\beta}. \tag{5.29}$$

Fix l and $c_7 \in (0, c_5)$ such that

$$l^{-\varkappa'+2} = \frac{c_5 - c_7}{c_6} E \tag{5.30}$$

in the case of Theorem 5.1 i), or

$$e^{-b_2 l^\beta} = \frac{c_5 - c_7}{c_6} E \tag{5.31}$$

in the case of Theorem 5.1 ii). Putting together (5.26) - (5.31), we conclude that (5.24), or, respectively, (5.25) will follow from the inequality

$$\sum_{\gamma\in\mathbb{Z}^2,\,|\gamma|\leq l} \omega_\gamma \leq c_7 E \tag{5.32}$$

provided that l satisfies (5.30) or, respectively, (5.31). Set

$$N(l) := \#\{\gamma \in \mathbb{Z}^2\,;\, |\gamma| \leq l\},$$

so that we have

$$N(l) = \pi l^2(1 + o(1)), \quad l \to \infty. \tag{5.33}$$

Evidently, the probability that (5.32) holds, is not less than the probability that $\omega_\gamma \leq c_7 E/N(l)$ for each $\gamma \in \mathbb{Z}^2$ such that $|\gamma| \leq l$. Since the random variables ω_γ are identically distributed and independent, the last probability is equal to $\mathbb{P}(\omega_0 \leq c_7 E/N(l))^{N(l)}$. Combining the above inequalities, and using the lower bound in (3.39), we get

$$\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq) \geq \frac{\text{area}\,\Lambda^*_{2L}}{(2\pi)^2}\, \mathbb{P}(\omega_0 \leq c_7 E/N(l))^{N(l)} - e^{-E^{-\eta}}, \tag{5.34}$$

where l is chosen to satisfy (5.30) with an arbitrary $\varkappa' \in (2, \varkappa)$ in the case of Theorem 5.1 i), or to satisfy (5.31) with an arbitrary fixed $b_2 < b_0$ in the case of Theorem 5.1 ii). Putting together (5.34), (2.1), (5.30), and (5.33), we get

$$\limsup_{E\downarrow 0} \frac{\ln|\ln(\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq))|}{|\ln E|} \leq \frac{2}{\varkappa' - 2}$$

for any $\varkappa' \in (2, \varkappa)$ such that $\eta > \frac{2}{\varkappa'-2}$. Letting $\varkappa' \uparrow \varkappa$, we get (5.1). Similarly, putting together (5.34), (2.1), (5.31), and (5.33), we get

$$\liminf_{E\downarrow 0} \frac{\ln(\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq))}{|\ln E|^{1+\frac{1}{\beta}}} \geq -\frac{\pi\kappa}{b_2}$$

for any $b_2 < b_0$. Letting

$$b_2 \uparrow b_0 = \begin{cases} \frac{1}{C} & \text{if} \quad \beta \in (0,2), \\ \frac{bC}{b+2C} & \text{if} \quad \beta = 2, \end{cases}$$

we get (5.2) – (5.3).

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